GLOBAL AND ASYMPTOTIC STABILITY OF ABSTRACT DIFFERENTIAL EQUATIONS AND OPERATOR-DIFFERENCE SCHEMES

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Abstract: *A priory estimates of global and asymptotical stability for Cauchy problems for abstract first and second order linear differential equations in Hilbert space are considered. A few scales of such type estimates are constructed in various energy norms. Analogous results are obtained for two- and three-level operator-difference schemes.*

Key words: *global and asymptotic stability, abstract differential equation, operator-difference scheme.*

1. INTRODUCTION

The notion of stability is a component of the correctness of a mathematical problem and in the general case it points to a continuous dependence of its solution u on input data of the problem φ, i.e., there exists such a constant $\rho > 0$, independent of the solution and input data, that for all $\varphi, \tilde{\varphi}$ from a certain allowable admissible set the following estimate holds:

$$\|\tilde{u} - u\|_1 \le \rho \|\tilde{\varphi} - \varphi\|_2, \tag{1}$$

where $\|\cdot\|_1$, $\|\cdot\|_2$ are certain norms and $\tilde{u}$ is the solution of the same problem with perturbed input data $\tilde{\varphi}$. The problem of stability becomes particularly urgent in the mathematical modelling of applied problems where input data can be given roughly (as a result of experimental measurements, observations etc.). For linear operator equations, estimate (1) is equivalent to the a priori estimate:

$$\|u\|_1 \le \rho \|\varphi\|_2 .$$

In the case of differential equations ρ most commonly take the following magnitudes:

$$\rho = e^{-Ct}, \quad \rho = C, \quad \rho = e^{Ct}; \quad C = const > 0,$$

where t is variable. The problem is called globally stable (or stable for a long time) if $\rho = \rho(t) \to const$ when $t \to +\infty$, and asymptotically stable if $\rho(t) \to 0$ when $t \to +\infty$.

Extensive literature is devoted to the construction of a priori estimates for linear evolutionary differential-operator equations (see e.g. [8-12]). In the present note we give a brief review of results obtained in [1-7].

2. ABSTRACT FIRST ORDER LINEAR CAUCHY PROBLEM

Let H be a real separable Hilbert space with an inner product $(\cdot,\cdot)$ and a norm $\|\cdot\|$, and A — an unbounded self-adjoint positively defined linear operator with the domain $D(A)$ dense in H. The expression $(u,v)_A = (Au,v),\ u,v \in D(A),$ satisfies the axioms of the inner product. The closure of $D(A)$ in the norm $\|u\|_A = (u,u)_A^{1/2}$ is so-called energy space $H_A \subset H$. In an analogous way on obtains the space $H_{A^{-1}} \supset H$. Further, $H_{A^{-1}} = H_A^*$ is the adjoint space for H_A, inner product (u,v) can be continuously extended on $H_{A^{-1}} \times H_A$, and the operator A can be extended to the mapping $A: H_A \to H_{A^{-1}}$. Spaces H_A, H and $H_{A^{-1}}$ form a Gelfand triple: $H_A \subset H \subset H_{A^{-1}}$.

We also introduce Lebesgue space $L_2(0,T;H)$ of functions $u(t)$ that map the segment $(a,b) \subset R$ to H [10,12], with inner product and norm:

$$(u,v)_{L_2(a,b;H)} = \int_a^b (u(t),v(t))\,dt, \quad \|u\|_{L_2(a,b;H)} = (u,u)_{L_2(a,b;H)}^{1/2} .$$

Consider the abstract Cauchy problem:

$$Bu'(t) + Au(t) = f(t), \ \ t > 0; \ \ u(0) = u_0; \qquad (\cdot)' = \frac{d}{dt}(\cdot), \qquad (2)$$

where B is self-adjoint positively defined linear operator in H and A is an unbounded self-adjoint positively defined linear operator in H_B.

2.1. Global Stability

Taking inner product of (2) with $2u$ we obtain:

$$2\,(Bu',u)+2\,(Au,u)=2\,(f,u).$$

From here, using Cauchy-Schwarz inequality follows:

$$\left(\|u(t)\|_B^2\right)'+2\,\|u(t)\|_A^2=2\left(f(t),u(t)\right)\le\|u(t)\|_A^2+\|f(t)\|_{A^{-1}}^2. \qquad (2)$$

Integrating (2) we finally obtain well known energy estimate:

$$\|u(t)\|_B^2+\int_0^t\|u(s)\|_A^2\,ds\le\|u_0\|_B^2+\int_0^t\|f(s)\|_{A^{-1}}^2\,ds. \qquad (3)$$

Using obvious relations

$$\left(\|u(t)\|_B^2\right)'=2\,\|u(t)\|_B\left(\|u(t)\|_B\right)', \qquad \|u(t)\|_A^2\ge\|u(t)\|_A\,\|u(t)\|_B$$

and Cauchy-Schwarz inequality

$$\left(f(t),u(t)\right)\le\|u(t)\|_B\,\|f(t)\|_{B^{-1}}.$$

from (2) also follows:

$$\|u(t)\|_B+\int_0^t\|u(s)\|_A\,ds\le\|u_0\|_B+\int_0^t\|f(s)\|_{B^{-1}}\,ds. \qquad (4)$$

By Fourier methods one obtains a priori estimates (see [4,7]):

$$\int_0^t\int_0^t\frac{\|u(s)-u(s')\|_B^2}{|s-s'|^2}ds\,ds'\le4\pi\left(\|u_0\|_B^2+\int_0^t\|f(s)\|_{A^{-1}}^2ds\right). \qquad (5)$$

From (3) and (5) one easily obtains the following scale of a priori estimates (see [4,7]):

$$\max_{s\in[0,t]}\|Bu(s)\|_{A^{-1}}^2+\int_0^t\|u(s)\|_B^2ds\le2\left(\|Bu_0\|_{A^{-1}}^2+\int_0^t\left\|A^{-1}f(s)\right\|_B^2ds\right),$$

$$\max_{s\in[0,t]}\|u(s)\|_B^2+\int_0^t\|u(s)\|_A^2ds+\int_0^t\int_0^t\frac{\|u(s)-u(s')\|_B^2}{|s-s'|^2}ds\,ds'\le(4\pi+2)\left(\|u_0\|_B^2+\int_0^t\|f(s)\|_{A^{-1}}^2ds\right)$$

$$\max_{s\in[0,t]}\|u(s)\|_A^2+\int_0^t\|Au(s)\|_{B^{-1}}^2ds+\int_0^t\|u'(s)\|_B^2ds\le3\left(\|u_0\|_A^2+\int_0^t\|f(s)\|_{B^{-1}}^2ds\right), \qquad (6)$$

$$\max_{s\in[0,t]}\|Au(s)\|_{B^{-1}}^2+\int_0^t\left\|B^{-1}Au(s)\right\|_A^2ds+\int_0^t\int_0^t\frac{\|u'(s)-u'(s')\|_B^2}{|s-s'|^2}ds\,ds'\le$$

$$\le (8\pi+2)\left(\|Au_0\|^2_{B^{-1}} + \int_0^t \|B^{-1}f(s)\|^2_A ds + \int_0^t\int_0^t \frac{\|f(s)-f(s')\|^2_{B^{-1}}}{|s-s'|^2}\, ds\, ds' \right),$$

etc.

Analogously, from (4) one obtains [7]:

$$\|u(t)\|_A + \int_0^t \|Au(s)\|_{B^{-1}} ds \le \|u_0\|_A + \int_0^t \|B^{-1}f(s)\|_A ds\,,$$

$$\|Au(t)\|_{B^{-1}} + \int_0^t \|B^{-1}Au(s)\|_A ds \le \|Au_0\|_{B^{-1}} + \int_0^t \|AB^{-1}f(s)\|_{B^{-1}} ds\,, \qquad (7)$$

$$\|B^{-1}Au(t)\|_A + \int_0^t \|AB^{-1}Au(s)\|_{B^{-1}} ds \le \|B^{-1}Au_0\|_A + \int_0^t \|B^{-1}AB^{-1}f(s)\|_A ds\,,$$

etc.

2.2. Asymptotic Stability

Using inequality

$$\|u\|^2_A \ge \lambda_1 \|u\|^2_B\,, \qquad (8)$$

where λ_1 is the minimal eigenvalue of the spectral problem $Au = \lambda Bu$, from (2) follows:

$$\left(\|u(t)\|^2_B\right)' + \lambda_1 \|u(t)\|^2_B \le \|f(t)\|^2_{A^{-1}} \qquad (9)$$

and

$$\left(\|u(t)\|_B\right)' + \lambda_1 \|u(t)\|_B \le \|f(t)\|_{B^{-1}}\,. \qquad (10)$$

From (9), similarly as in the previous case one obtains (see [7]):

$$\|Bu(t)\|^2_{A^{-1}} \le e^{-\lambda_1 t}\left(\|Bu_0\|^2_{A^{-1}} + \int_0^t e^{\lambda_1 s}\|A^{-1}f(s)\|^2_B ds \right),$$

$$\|u(t)\|^2_B \le e^{-\lambda_1 t}\left(\|u_0\|^2_B + \int_0^t e^{\lambda_1 s}\|f(s)\|^2_{A^{-1}} ds \right),$$

$$\|u(t)\|^2_A \le e^{-\lambda_1 t}\left(\|u_0\|^2_A + \int_0^t e^{\lambda_1 s}\|f(s)\|^2_{B^{-1}} ds \right), \qquad (11)$$

$$\|Au(t)\|^2_{B^{-1}} \le e^{-\lambda_1 t}\left(\|Au_0\|^2_{B^{-1}} + \int_0^t e^{\lambda_1 s}\|B^{-1}f(s)\|^2_A ds \right),$$

etc.

From (10) one obtains (see [7]):

$$\|u(t)\|_B \le e^{-\lambda_1 t}\left(\|u_0\|_B + \int_0^t e^{\lambda_1 s}\,\|f(s)\|_{B^{-1}} ds\right),$$

$$\|u(t)\|_A \le e^{-\lambda_1 t}\left(\|u_0\|_A + \int_0^t e^{\lambda_1 s}\,\left\|B^{-1} f(s)\right\|_A ds\right),$$

$$\|Au(t)\|_{B^{-1}} \le e^{-\lambda_1 t}\left(\|Au_0\|_{B^{-1}} + \int_0^t e^{\lambda_1 s}\,\left\|AB^{-1} f(s)\right\|_{B^{-1}} ds\right), \tag{12}$$

$$\left\|B^{-1}Au(t)\right\|_A \le e^{-\lambda_1 t}\left(\left\|B^{-1}Au_0\right\|_A + \int_0^t e^{\lambda_1 s}\,\left\|B^{-1}AB^{-1} f(s)\right\|_A ds\right),$$

etc.

3. ABSTRACT SECOND ORDER LINEAR CAUCHY PROBLEM

Consider the abstract Cauchy problem:

$$Du''(t) + Bu'(t) + Au(t) = f(t),\ \ t > 0; \qquad u(0) = u_0; \qquad u'(0) = u_1\ , \tag{13}$$

where D is self-adjoint positively defined linear operator in H, A is unbounded self-adjoint positively defined linear operator in H_D and B is nonnegative linear operator in H. Using energy method and Gronwall lemma one obtains the following standard a priori estimate [12]:

$$\|u(t)\|_D^2 + \|u'(t)\|_A^2 \le e^t\left(\|u_1\|_D^2 + \|u_0\|_A^2 + \int_0^t \|f(s)\|_{D^{-1}}^2 ds\right).$$

3.1. Global Stability

Taking inner product of (13) with 2u' and using Cauchy-Schwarz inequality we obtain:

$$\left(\|u'\|_D^2 + \|u\|_A^2\right)' + 2\,\|u'\|_B^2 = 2\,(f,u) \le 2\,\|u'\|_D\,\|f\|_{D^{-1}} \le \left(\|u'\|_D^2 + \|u\|_A^2\right)^{1/2}\,\|f\|_{D^{-1}}$$

.

After integration one obtains [2,7]:

$$\left(\|u(t)\|_D^2 + \|u'(t)\|_A^2\right)^{1/2} \le \left(\|u_1\|_D^2 + \|u_0\|_A^2\right)^{1/2} + \int_0^t \|f(s)\|_{D^{-1}} ds\,. \tag{14}$$

If $B = 0$ from (14) one obtains the following scale of a priori estimates [7]:

$$\max_{s\in[0,t]}\|u(s)\|_D \le \|u_0\|_D + \|Du_1\|_{A^{-1}} + \int_0^t \|f(s)\|_{A^{-1}}\,ds,$$

$$\max_{s\in[0,t]}\left(\|u'(s)\|_D + \|u(s)\|_A\right) \le \sqrt{2}\left(\|u_1\|_D + \|u_0\|_A + \int_0^t \|f(s)\|_{D^{-1}}ds\right),$$

$$\max_{s\in[0,t]}\left(\|u''(s)\|_D + \|u'(s)\|_A + \|Au(s)\|_{D^{-1}}\right) \le$$

$$\le \left(\sqrt{2}+1\right)\left(\|u_1\|_A + \|Au_0\|_{D^{-1}} + \|f(0)\|_{D^{-1}} + \int_0^t \|f'(s)\|_{D^{-1}}ds + \int_0^t \left\|D^{-1}f(s)\right\|_A ds\right), \quad (15)$$

$$\max_{s\in[0,t]}\left(\|u'''(s)\|_D + \|u''(s)\|_A + \|Au'(s)\|_{D^{-1}} + \left\|D^{-1}Au(s)\right\|_A\right) \le$$

$$\le 2\sqrt{2}\left(\|Au_1\|_{D^{-1}} + \left\|D^{-1}Au_0\right\|_A + \|f'(0)\|_{D^{-1}} + \left\|D^{-1}f(0)\right\|_A + \int_0^t \|f''(s)\|_{D^{-1}}ds + \int_0^t \left\|AD^{-1}f(s)\right\|_{D^{-1}}ds\right)$$

etc.

3.2. Asymptotic Stability

Consider the homogeneous Cauchy problem:

$$u''(t) + Bu'(t) + Au(t) = 0,\ \ t > 0;\ \ u(0) = u_0;\quad u'(0) = u_1\ , \quad (16)$$

where A and B are unbounded linear positive self-adjoint operators in H. Assume also that $AB = BA$ where this product is defined.

Let the operator inequality $A - 0.25B^2 \ge c_0 A,\quad 0 < c_0 < 1$, is satisfied. Taking inner product of (16) with 2u' one obtains:

$$\left(\|u'\|^2 + \|u\|_A^2\right)' + 2\|u'\|_B^2 = 0. \quad (17)$$

Identity (17) can be rearranged in the following manner

$$\left(\left\|u' + \frac{1}{2}Bu\right\|^2 + \|u\|^2_{A-\frac{1}{4}B^2}\right)' + \left\|u' + \frac{1}{2}Bu\right\|_B^2 + \|u\|^2_{B(A-\frac{1}{4}B^2)} = 0\ . \quad (18)$$

Using inequality

$$\|u\|_B^2 \ge \lambda_1 \|u\|^2,$$

where λ_1 is the minimal eigenvalue of the spectral problem $Bu = \lambda u$, and integrating (18) we obtain a priori estimate:

$$\left\|u'(t) + \frac{1}{2}Bu(t)\right\|^2 + \|u(t)\|^2_{A-\frac{1}{4}B^2} \le e^{-\lambda_1 t}\left(\left\|u_1 + \frac{1}{2}Bu_0\right\|^2 + \|u_0\|^2_{A-\frac{1}{4}B^2}\right). \quad (19)$$

Similarly, from (17) follows

$$\left(\left\| Au + \frac{1}{2} Bu' \right\|^2_{(A-\frac{1}{4}B^2)^{-1}} + \|u'\|^2 \right)' + \left\| Au + \frac{1}{2} Bu' \right\|^2_{B(A-\frac{1}{4}B^2)^{-1}} + \|u'\|^2_B = 0\,,$$

and after integration:

$$\left\| Au(t) + \frac{1}{2} Bu'(t) \right\|^2_{(A-\frac{1}{4}B^2)^{-1}} + \|u'(t)\|^2 \le e^{-\lambda_1 t} \left(\left\| Au_0 + \frac{1}{2} Bu_1 \right\|^2_{(A-\frac{1}{4}B^2)^{-1}} + \|u_1\|^2 \right) \tag{20}$$

From (19) and (20) one obtains the following estimate of the asymptotic stability [7,2]:

$$\|u'(t)\|^2 + \|u(t)\|^2_A \le \frac{4}{c_0}\, e^{-\mu_1 t} \left(\|u_1\|^2 + \|u_0\|^2_A \right). \tag{21}$$

In the case when the operator inequality $0.25B^2 - A \ge c_1 B^2, \quad 0 < c_1 < 0.25$, is fulfilled an analogous a priori estimate holds [7,2]:

$$\|u'(t)\|^2 + \|Bu(t)\|^2 \le \frac{2}{c_1}\, e^{-\nu_1 t} \left(\|u_1\|^2 + \|Bu_0\|^2 \right). \tag{22}$$

Here $\nu_1 > 0$ is the first eigenvalue of the operator $B - (B^2 - 4A)^{1/2}$.

4. OPERATOR-DIFFERENCE SCHEMES

Analogous results hold for two- and three-level operator-difference schemes in a Hilbert space H (see [1-4]).

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