HOMOGENEOUS SYSTEM OF DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS OF SYMMETRIC MATRIX

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In this paper is solved one characteristic problem of system of differential equations with constant coefficients of symmetric matrix. Observed problem is appendix to the well-known books [1,2,3].

1. Formulation of the problem

Solve the system

$$\frac{dX}{dt} = AX \tag{1}$$

of differential equations, where

$$\frac{dX}{dt} = \left[\frac{dx_i}{dt}\right]_{n\times 1}, \qquad X = [x_i]_{n\times 1}, \qquad (1 \le i \le n)$$

and $A = [a_{ij}]_{n\times n}$ is the symmetric matrix with elements

1. $a_{ij} = i(n+1-j), \qquad (i \le j) \qquad \text{and} \qquad a_{ij} = a_{ji}$
2. $a_{ij} = \sum\limits_{r=j}^{n} r^{-1}, \qquad (i \le j)$ and $a_{ij} = a_{ji}$.

2. Solution of the problem

1. Since the inverse matrix $B = A^{-1}$ of the matrix A is a particular Jacobi matrix, the eigenvalues of the matrix $B = [b_{ij}]$ can easily be determined where

$$(n+1)b_{ij} = 2, \qquad (i = j)$$

$$(n+1)b_{ij} = -1, \qquad (i = j \pm 1)$$

$$(n+1)b_{ij} = 0, \qquad (|\, i - j\,| > 1).$$

Now let $P_n(\lambda) = 0$ be the characteristic polynomial of the matrix $(n+1)[b_{ij}]_{n\times n}$. From the characteristic polynomial of the matrix $(n+1)[b_{ij}]_{n\times n}$ we obtain the following recurrent formula

$$P_1(\lambda) = 2 - \lambda, \qquad P_2(\lambda) = 3 - 4\lambda + \lambda^2, \qquad P_k(\lambda) = (2-\lambda)P_{k-1}(\lambda) - P_{k-2}(\lambda).$$

By solving this difference equation, we obtain

$$P_n(\lambda) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta},$$

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where α and β are the roots of the equation

$$x^2 - (2-\lambda)x + 1 = 0.$$

Hence $\epsilon = \frac{\alpha}{\beta}$, where $\epsilon^{n+1} = 1$ and $\alpha \neq 1$, is a root of the equation $P_n(\lambda) = 0$ and all different roots of P_n are

$$\lambda_k = \frac{4}{n+1} \sin^2 \frac{k\pi}{2(n+1)}, \qquad (1 \leq k \leq n)$$

and the eigenvalues of the matrix A are $\mu_k = \lambda_k^{-1}$, i.e.

$$\mu_k = \frac{n+1}{4\sin^2 \frac{k\pi}{2(n+1)}}, \qquad (1 \leq k \leq n)$$

which correspond to the following eigenvectors

$$X_k = \left[\sin \frac{k\pi}{n+1}, \sin \frac{2k\pi}{n+1}, \cdots, \sin \frac{nk\pi}{n+1}\right]^T, \qquad (1 \leq k \leq n).$$

Now the general solution of (1) is given by

$$X = \sum_{k=1}^{n} C_k X_k e^{\mu_k},$$

where C_k, $(1 \leq k \leq n)$ are arbitrary constants. It completes this case.

2. If from each column of the matrix $\lambda A - I$ is subtracted the next one, and then if from each row is subtracted the next one, we obtain

$$P_n(\lambda) = |\,\lambda A - I\,| = \begin{vmatrix} \lambda-2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & \frac{\lambda}{2}-2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \frac{\lambda}{3}-2 & \cdots & 0 & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & \cdots & \frac{\lambda}{n-1}-2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & \frac{\lambda}{n}-1 \end{vmatrix}.$$

Let us denote

$$F_n(\lambda) = \begin{vmatrix} \lambda-2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & \frac{\lambda}{2}-2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \frac{\lambda}{3}-2 & \cdots & 0 & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & \cdots & \frac{\lambda}{n-1}-2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & \frac{\lambda}{n}-2 \end{vmatrix},$$

we obtain

$$P_n(\lambda) = F_n(\lambda) + F_{n-1}(\lambda),$$

$$F_n(\lambda) = (\frac{\lambda}{n} - 2)F_{n-1}(\lambda) - F_{n-2}(\lambda).$$

By elimination of F, we obtain

$$nP_n(\lambda) = (\lambda - 2n + 1)P_{n-1}(\lambda) - (n-1)P_{n-2}(\lambda).$$

Since $P_0(\lambda) = 1 = L_0(\lambda)$ and $P_1(\lambda) = \lambda - 1 = -L_1(\lambda)$, then from the previous equation it follows

$$P_n(\lambda) = (-1)^n L_n(\lambda),$$

where $L_n(\lambda) = \sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\lambda^k}{k!}$ is the Laguerre polynomial of n-th degree. Thus the eigenvalues $\lambda_k, (1 \le k \le n)$ of the matrix A are the zeros of the Laguerre polynomial $L_n(\lambda)$.

The coefficients a_{ij} satisfy

$$i(a_{ij} - a_{i+1,j}) = \begin{cases} 0, & (1 \le i < j \le n) \\ 1, & (1 \le j \le i \le n-1) \end{cases},$$

$$n a_{nj} = 1, (1 \le j \le n).$$

On the other hand, for $1 \le j \le n$, it holds

$$(2i-1)a_{ij} - i a_{i+1,j} - (i-1)a_{i-1,j} = \delta_{ij}, \qquad (1 \le i \le n) \tag{2}$$

where $a_{0j} = a_{n+1,j} = 0$. For $\lambda \ne 0$ it will exist eigenvector $X = x^T = [x_1, x_2, \cdots, x_n]^T$.

Since $x_i = \mu \sum\limits_{j=1}^{n} a_{ij} x_j$, by putting $\mu = \lambda^{-1}$, from (2) we obtain

$$(2i-1)x_i - i x_{i+1} - (i-1)x_{i-1} = \mu x_i, \qquad (1 \le i \le n) \tag{3}$$

where $x_0 = x_{n+1} = 0$.

If $x_1 = 0$, from the previous equationa we obtain recursively $x_2 = x_3 = \cdots = x_n = 0$, which is a contradiction because the eigenvector is a nonzero vector. Hence $x_1 \ne 0$. Since the eigenvector is determined up to a nonzero scalar, without loss of generality we assume that $x_1 = 1$. Let be $x_1 = 1$, then it will be $x_{i+1} = L_i(\mu)$, $(1 \le i \le n-1)$ such that for the Laguerre polynomial $L_r(z)$ it holds

$$L_0(z) = 1, \qquad L_1(z) = 1 - z,$$

$$r L_r(z) = (2r - 1 - z)L_{r-1}(z) - (r-1)L_{r-2}(z).$$

For the coordinates of the eigenvector X, which are determined from (3), it will hold $L_n(\mu) = 0$, which means that μ is a root of $L_n(\mu) = 0$. The polynomial equation $L_n(\mu) = 0$ has $\mu_k, (1 \le k \le n)$ positive roots. Hence, the eigenvalues of the matrix A are $\lambda_k = \mu_k^{-1}, (1 \le k \le n)$ which correspond to the eigenvectors

$$X_k = [L_0(\mu_k), L_1(\mu_k), \cdots, L_{n-1}(\mu_k)]^T, \qquad (1 \le k \le n).$$

At the end, the general solution of the matrix equation (1) is

$$X = \sum_{k=1}^{n} C_k X_k e^{\lambda_k},$$

where $C_k, (1 \le k \le n)$ are arbitrary constants, and that finishes this case.

R E F E R E N C E S

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