ONE EXAMPLE OF INTERPOLATION IN M SPACES USING BLASCHKE PRODUCTS

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Abstract

We give a solution of interpolation problem in M spaces.

Let D be the unit disk. We denote by M the space of holomorphic function in D such that

$$\int_0^{2\pi} \log^+ Mf(\theta) d\theta < \infty$$

where

$$Mf(\theta) = \sup_{0 \le r < 1} |f(re^{i\theta})|$$

and

$$\log^+ a = \begin{cases} 0, & 0 < a \le 1 \\ \log a, & a \ge 1 \end{cases}$$

(see [1]).

The next statements are valid: $\bigcup\limits_{p>0} H^p \subseteq M \subseteq N^+ \subseteq N$ (see [1]). All the inclusions are proper. For the first inclusion in [1] is given an example of the function which is in M and it is not in $\bigcup\limits_{p>0} H^p$. This function is defined with

$$f(z) = \exp\left(\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log \psi(t) \frac{dt}{2\pi}\right) \qquad (1)$$

where

$$\psi(t) = \begin{cases} \exp \frac{1}{|\theta|}, & |\theta| \le 1 \\ e, & 1 \le |\theta| \le \pi \end{cases}$$

Here we solve the next interpolation problem. Let $(\lambda_k)_{k=1}^{\infty}$ be a sequence in D and $(c_k)_{k=1}^{\infty}$ be a given number sequence. We look for conditions for these sequences $(\lambda_k)_{k=1}^{\infty}$ and $(c_k)_{k=1}^{\infty}$ such that exists a function $f \in M$ so that $f(\lambda_k) = c_k$, $k = 1, 2, \ldots$.

We will show that this condition

$$\sum_{k=1}^{\infty}(1-|\lambda_k|)<\infty \qquad (2)$$

is necessary.

Let the condition is not satisfied by the sequence $(\lambda_k)_{k=1}^{\infty}$ i.e $\sum_{k=1}^{\infty}(1-|\lambda_k|)=\infty$, and c_k are the values of some function of the class M in λ_k. It is known that the zeros of every function from the N class must satisfy (2)(see [4],p 333). So there is $c_p \neq 0$. Let $c_p' = c_p e^{i\varphi}$ where $\varphi \in (0, 2\pi)$ is fixed. Then there is not function g of the class M such that

$$g(\lambda_k)=\begin{cases} c_k, & k\neq p \\ c_k', & k=p \end{cases}$$

because if this is not so, the function $h(z)=f(z)-g(z)$ is in the M class and theirs zeros will not satisfy the condition (2).

We denote by $b(z)$ the Blaschke product

$$b(z)=\prod_{j=1}^{\infty}\frac{\lambda_j-z}{1-\overline{\lambda_j}z}\frac{|\lambda_j|}{\lambda_j}, \quad \sum_{j=1}^{\infty}(1-|\lambda_j|)<\infty,\ z\in D$$

and

$$b_k(z)=\prod_{j\neq k}\frac{\lambda_j-z}{1-\overline{\lambda_j}z}\frac{|\lambda_j|}{\lambda_j}.$$

We will use the next theorem.

Theorem A [5]p. 69: *The sum of a uniformly convergent series of holomorphic functions is a holomorphic function in every inner point on the set where the series uniformly converges.*

Theorem 1: *Let* $(\lambda_k)_{k=1}^{\infty}$ *is a sequence in* D *such that* $\sum_{j=1}^{\infty}(1-|\lambda_j|)<\infty$ *and* $(c_k)_{k=1}^{\infty}$ *is a sequence such that* $\sum_{j=1}^{\infty}\left|\frac{c_j}{b_j(\lambda_j)}\right|<\infty$. *Then the interpolation problem* $f(\lambda_j)=c_j$ *has a solution in* M.

Proof: We define the function

$$f(z)=g(z)\sum_{k=1}^{\infty}c_k\frac{b_k(z)}{b_k(\lambda_k)}\frac{1}{g(\lambda_k)}$$

where $g(z)$ is the function (1). It is clear that $f(\lambda_k)=c_k$, $k=1,2,\ldots$. We will show that $f\in M$. First of all we will show that f is holomorphic. It is true that $|g(\lambda_k)|\geq 1$.

$$|g(\lambda_k)|=\exp\left(Re\int_{-\pi}^{\pi}\frac{e^{it}+\lambda_k}{e^{it}-\lambda_k}\log\psi(t)dt\right).$$

Because $Re\left(\frac{e^{it}+z}{e^{it}-z}\right) = \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} > 0$, where $z = re^{i\theta} \in D$ and $\log(\psi(\theta))$ is nonnegative it follows that $|g(\lambda_k)| \geq 1$. In the following we will use the next property of Blachke products: If $b(z)$ is Blachke product in D then for every $z \in D$ it is true $|b(z)| \leq 1$ ([3] p. 86).

It is true that

$$|f(z)| \leq \sum_{k=1}^{\infty} \left|\frac{c_k}{b_k(\lambda_k)}\right| \frac{|b_k(z)|}{g(\lambda_k)} \leq \sum_{k=1}^{\infty} \left|\frac{c_k}{b_k(\lambda_k)}\right| = K < \infty.$$

Because of the criteria of Weierstrass, for uniform convergence follows that the functional series $\sum_{k=1}^{\infty} c_k \frac{b_k(z)}{b_k(\lambda_k)} \frac{1}{g(\lambda_k)}$ converges uniformly on D, and by Theorem A, the sum of the series is holomorphic function $h(z)$ so that $|h(z)| \leq \sum_{k=1}^{\infty} \left|\frac{c_k}{b_k(\lambda_k)}\right| = K$ for all $z \in D$ holds, then $h \in H^{\infty}$ and $\|h\|_{\infty} \leq K$. We will show that $f \in M$. It is true

$$\log^+ ab \leq \log^+ a + \log^+ b.$$

Now

$$Mf(\theta) = \sup_{0\leq r\leq 1} |f(re^{i\theta})| = \sup_{0\leq r \leq 1} \leq$$

$$\leq K \sup_{0\leq r\leq 1} |g(re^{i\theta})| = KMg(\theta).$$

So $\log^+ Mf(\theta) \leq (KMg(\theta)) \leq \log^+ K + \log^+ Mg(\theta)$. Finally

$$\int_0^{2\pi} \log^+ Mf(\theta)d\theta \leq 2\pi \log^+ K + \int_0^{2\pi} \log^+ Mg(\theta)d\theta < \infty$$

i.e $f \in M$.

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РЕЗИМЕ

Во овој труд е даден доволен услов за интерполација во единичниот диск со функции од класата M.

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