A Proof of SWSE Method

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Abstract

In the recent paper [3] is given a method for calculation of the solutions of the analytical system of differential equations

$$\dot{x}(t) = f(t, x(t)) \ , \quad x(t) \in \mathbf{R}^n \ , \quad x(0) = x^0$$

with the unknown variable $x(t)$, called SWSE (Summing Weighted Sequential Errors) method. The solutions are presented as functional series. In this paper we give another proof of the convergence of that functional series.

Keywords and phrases: System of ordinary differential equations, SWSE method, functional series.

1 Main results

In [1], a functional expansion is applied to obtain the flow of the nonstationary analytic vector field $X_t(x)$, i.e. the solution of

$$\frac{\partial}{\partial t}\Phi_{t_0,t}(x) = X_t(\Phi_{t_0,t}(x)) \tag{1.1}$$

satisfying $\Phi_{t_0,t_0} = \mathrm{Id}$, where Id is the identity operator. The equation (1.1), is solved by the following "chronological" formal series

$$\Phi_{t_0,t} = \mathrm{Id} + \sum_{m=1}^{\infty} \int_{t_0}^{t} \mathrm{d}\tau_1 \int_{t_0}^{\tau_1} \mathrm{d}\tau_2 \cdots \int_{t_0}^{\tau_{m-1}} \mathrm{d}\tau_m \cdot X_{\tau_m} \circ \cdots \circ X_{\tau_1} \tag{1.2}$$

where by $\circ$ we denote the composition of mappings.

Consider the system of ordinary differential equation (ODE)

$$\dot{x}(t) = f(t, x(t)) \ , \quad x \in \mathbf{R}^n \tag{1.3}$$

where the elements f_s of the function f have a Laurent's expansion for all t in a neighborhood U of $t = t_0$

$$f_s(t, z) = \sum_{i_1,\ldots,i_n \in \mathbf{Z}} f_{s i_1 \ldots i_n}(t) \cdot z_1^{i_1} \cdot z_2^{i_2} \cdots z_n^{i_n} \ , \quad s = 1, \ldots n \ , \quad z \in W \subseteq \mathbf{R}^n \ . \tag{1.4}$$

In general, it is possible $0 \notin W$. (If $0 \in W$ then the Laurent's series reduces to the Taylor's series.) We suppose that all functions $f_{s i_1 \ldots i_n}(t)$ are regular in U, and that the convergence in (1.4) is uniform in each closed subset of $U \times W$. The Laurent's series (1.4) is introduced for proving the main results only. In the main results, contained in Sections 2, 3 and 4, the Laurent's series does not appear.

The initial conditions $x_i(t_0) = C_i,\ i = 1, \ldots, n$ are such that $(C_1, \ldots, C_n)$ belongs to the domain W of the convergence of the Laurent's series (1.4).

To solve (1.3), let us introduce the following functions

$$x_{i_1 i_2 \ldots i_n} = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad (i_1, \ldots, i_n \in \mathbf{Z}) \tag{1.5}$$

so that $x_1 = x_{10\ldots0}\ , \ldots ,\ x_n = x_{0\ldots01}$. We have

$$\begin{aligned}
\dot{x}_{i_1 i_2 \ldots i_n} &= \frac{\mathrm{d}}{\mathrm{d}t}(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}) = \\
&= i_1 x_{(i_1-1) i_2 \ldots i_n} \dot{x}_1 + \cdots + i_n x_{i_1 i_2 \ldots (i_n-1)} \dot{x}_n = \\
&= i_1 x_{(i_1-1) i_2 \ldots i_n} \sum_{p_1, \ldots, p_n \in \mathbf{Z}} f_{1 p_1 \ldots p_n} x_{p_1 p_2 \ldots p_n} + \cdots \\
&\quad + i_n x_{i_1 i_2 \ldots (i_n-1)} \sum_{p_1, \ldots, p_n \in \mathbf{Z}} f_{n p_1 \ldots p_n} x_{p_1 p_2 \ldots p_n} = \\
&= i_1 \sum_{p_1, \ldots, p_n \in \mathbf{Z}} f_{1 p_1 \ldots p_n} x_{(i_1+p_1-1)(i_2+p_2)\ldots(i_n+p_n)} + \cdots \\
&\quad + i_n \sum_{p_1, \ldots, p_n \in \mathbf{Z}} f_{n p_1 \ldots p_n} x_{(i_1+p_1)(i_2+p_2)\ldots(i_n+p_n-1)} = \\
&= \sum_{s=1}^{n} i_s \sum_{p_1, \ldots, p_n \in \mathbf{Z}} f_{s p_1 \ldots p_n} x_{(i_1+p_1)\ldots(i_s+p_s-1)\ldots(i_n+p_n)} = \\
&= \sum_{s=1}^{n} i_s \sum_{j_1, \ldots, j_n \in \mathbf{Z}} f_{s (j_1-i_1)\ldots(j_s-i_s+1)\ldots(j_n-i_n)} x_{j_1 \ldots j_n}\ ,
\end{aligned}$$

i.e.

$$\dot{x}_{i_1 \ldots i_n} = \sum_{j_1, \ldots, j_n \in \mathbf{Z}} h_{i_1 \ldots i_n j_1 \ldots j_n} \cdot x_{j_1 \ldots j_n} \tag{1.6}$$

for $i_1, \ldots, i_n \in \mathbf{Z}$, and where

$$h_{i_1 \ldots i_n j_1 \ldots j_n} \stackrel{\mathrm{def}}{=} \sum_{s=1}^{n} i_s f_{s (j_1-i_1)\ldots(j_s-i_s+1)\ldots(j_n-i_n)}\ . \tag{1.7}$$

We have obtained the system (1.6), which is a linear system of differential equations with infinitely unknown functions $x_{i_1 \dots i_n},\ i_1, \dots, i_n \in \mathbf{Z}$. In [3] we obtain its solution by SWSE method. At first, in [3], we introduce the functions

$$P^{<k>}_{i_1 i_2 \dots i_n j_1 j_2 \dots j_n}(t) \quad (k \in \mathbf{N}_0,\ i_1, \cdots, i_n,\ j_1, \cdots, j_n \in \mathbf{Z})$$

as follows

$$P^{<0>}_{i_1 i_2 \dots i_n j_1 j_2 \dots j_n} = \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_n j_n},$$

$$P^{<k+1>}_{i_1 \dots i_n j_1 \dots j_n} = \frac{\mathrm{d}}{\mathrm{d}t} P^{<k>}_{i_1 \dots i_n j_1 \dots j_n} - \sum_{p_1, \dots, p_n \in \mathbf{Z}} h_{i_1 \dots i_n p_1 \dots p_n} \cdot P^{<k>}_{p_1 \dots p_n j_1 \dots j_n}\ . \tag{1.8}$$

where δ_{ij} is the Kronecker delta function. In [3] is proved that the functions $P^{<k>}_{i_1 \dots i_n j_1 \dots j_n},\ k = 0, 1, \dots$ are well defined.

Now we give another proof of the following theorem.

Theorem 1.1. *The solution of the system (1.3) of non-linear differential equations with the initial conditions $x_i(t_0) = C_i$ $(1 \le i \le n)$ in a neighborhood of $t = t_0$ is given by*

$$x_1(t) = \sum_{k=0}^{\infty} \frac{(t_0 - t)^k}{k!} \sum_{j_1, \dots, j_n \in \mathbf{Z}} P^{<k>}_{10\dots0 j_1 \dots j_n}(t) C_1^{j_1} C_2^{j_2} \cdots C_n^{j_n}$$

$$x_2(t) = \sum_{k=0}^{\infty} \frac{(t_0 - t)^k}{k!} \sum_{j_1, \dots, j_n \in \mathbf{Z}} P^{<k>}_{01\dots0 j_1 \dots j_n}(t) C_1^{j_1} C_2^{j_2} \cdots C_n^{j_n} \tag{1.9}$$

$$\vdots$$

$$x_n(t) = \sum_{k=0}^{\infty} \frac{(t_0 - t)^k}{k!} \sum_{j_1, \dots, j_n \in \mathbf{Z}} P^{<k>}_{0\dots01 j_1 \dots j_n}(t) C_1^{j_1} C_2^{j_2} \cdots C_n^{j_n}$$

where the functions $P^{<k>}_{i_1 \dots i_n j_1 \dots j_n}(t)$ are defined by (1.8) and (1.7).

Proof. A proof is given in [3]. In the following text we present a simpler proof. We prove that a solution of (1.6), with the initial conditions $x_{i_1 \dots i_n}(t_0) = C_1^{i_1} \cdots C_n^{i_n}$, is given by summing weighted sequential errors (SWSE) of the linear system (1.6) with infinitely many unknowns, i.e.

$$x_{i_1 \dots i_n}(t) = \sum_{k=0}^{\infty} \frac{(t_0 - t)^k}{k!} \sum_{j_1, \dots, j_n \in \mathbf{Z}} P^{<k>}_{i_1 \dots i_n j_1 \dots j_n}(t) C_1^{j_1} \cdots C_n^{j_n} \tag{1.10}$$

where the functions $P^{<k>}_{i_1 \dots i_n j_1 \dots j_n}(t)$ are given by (1.8) and (1.7). Specially, if

$$(i_1, \dots, i_n) \in \{(1, 0, \dots, 0),\ (0, 1, \dots, 0), \dots, (0, \dots, 0, 1)\}$$

we obtain the required solution (1.9).

For $i_1,\ldots,i_n,j_1,\ldots,j_n\in\mathbf{Z}$, let $\Phi_{i_1\ldots i_n j_1\ldots j_n}(t,t_0)$ be a function that satisfies

$$\Phi_{i_1\ldots i_n j_1\ldots j_n}(t_0,t_0)=\delta_{i_1 j_1}\cdots\delta_{i_n j_n} \tag{1.11}$$

and

$$\frac{\partial\Phi_{i_1\ldots i_n j_1\ldots j_n}(t,t_0)}{\partial t}=\sum_{\alpha_1,\ldots,\alpha_n\in\mathbf{Z}} h_{i_1\ldots i_n\alpha_1\ldots\alpha_n}(t)\cdot\Phi_{\alpha_1\ldots\alpha_n j_1\ldots j_n}(t,t_0) \tag{1.12}$$

where the functions $h_{i_1\ldots i_n\alpha_1\ldots\alpha_n}(t)$ are defined in (1.7).

Let us introduce the following functions

$$p^{<k>}_{i_1\ldots i_n}(t,z)=\sum_{j_1,\ldots,j_n\in\mathbf{Z}} P^{<k>}_{i_1\ldots i_n j_1\ldots j_n}(t)\cdot z_1^{j_1}\cdots z_n^{j_n}$$

The functions $p^{<k>}_{i_1\ldots i_n}(t,z)$ satisfy

$$\frac{\partial p^{<k>}_{i_1\ldots i_n}(t,z)}{\partial t}=\sum_{j_1,\ldots,j_n\in\mathbf{Z}}\frac{\mathrm{d}P^{<k>}_{i_1\ldots i_n j_1\ldots j_n}(t)}{\mathrm{d}t}\cdot z_1^{j_1}\cdots z_n^{j_n}=$$

$$=\sum_{j_1,\ldots,j_n\in\mathbf{Z}} P^{<k+1>}_{i_1\ldots i_n j_1\ldots j_n}(t)\cdot z_1^{j_1}\cdots z_n^{j_n}+$$

$$+\sum_{j_1,\ldots,j_n\in\mathbf{Z}}\ \sum_{\alpha_1,\ldots,\alpha_n\in\mathbf{Z}} h_{i_1\ldots i_n\alpha_1\ldots\alpha_n}\cdot P^{<k>}_{\alpha_1\ldots\alpha_n j_1\ldots j_n}\cdot z_1^{j_1}\cdots z_n^{j_n}=$$

$$=p^{<k+1>}_{i_1\ldots i_n}+\sum_{\alpha_1,\ldots,\alpha_n\in\mathbf{Z}} h_{i_1\ldots i_n\alpha_1\ldots\alpha_n}\sum_{j_1,\ldots,j_n\in\mathbf{Z}} P^{<k>}_{\alpha_1\ldots\alpha_n j_1\ldots j_n}\cdot z_1^{j_1}\cdots z_n^{j_n}=$$

$$=p^{<k+1>}_{i_1\ldots i_n}+\sum_{\alpha_1,\ldots,\alpha_n\in\mathbf{Z}} h_{i_1\ldots i_n\alpha_1\ldots\alpha_n}\cdot p^{<k>}_{\alpha_1\ldots\alpha_n}$$

Therefore

$$p^{<k+1>}_{i_1\ldots i_n}(t,z)=\frac{\partial p^{<k>}_{i_1\ldots i_n}(t,z)}{\partial t}-\sum_{\alpha_1,\ldots,\alpha_n\in\mathbf{Z}} h_{i_1\ldots i_n\alpha_1\ldots\alpha_n}(t)\cdot p^{<k>}_{\alpha_1\ldots\alpha_n}(t,z) \tag{1.13}$$

Let us introduce the following functions

$$q^{<k>}_{i_1\ldots i_n}(\tau,\tau_0,z)=\sum_{j_1,\ldots,j_n\in\mathbf{Z}}\Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)\cdot p^{<k>}_{j_1\ldots j_n}(\tau,z) \tag{1.14}$$

Using (1.13), we obtain

$$\frac{\partial q^{<k>}_{i_1\ldots i_n}(\tau,\tau_0,z)}{\partial\tau}=\sum_{j_1,\ldots,j_n\in\mathbf{Z}}\frac{\partial\Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)}{\partial t_0}\cdot p^{<k>}_{j_1\ldots j_n}(\tau,z)+$$

$$+\sum_{j_1,\ldots,j_n\in\mathbf{Z}} \Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)\cdot\frac{\partial p^{<k>}_{j_1\ldots j_n}(\tau,z)}{\partial\tau}=$$

$$=\sum_{j_1,\ldots,j_n\in\mathbf{Z}} \frac{\partial\Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)}{\partial t_0}\cdot p^{<k>}_{j_1\ldots j_n}(\tau,z)+$$

$$+\sum_{j_1,\ldots,j_n\in\mathbf{Z}} \Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)\sum_{\alpha_1,\ldots,\alpha_n\in\mathbf{Z}} h_{j_1\ldots j_n\alpha_1\ldots\alpha_n}(\tau)\cdot p^{<k>}_{\alpha_1\ldots\alpha_n}(\tau,z)+$$

$$+\sum_{j_1,\ldots,j_n\in\mathbf{Z}} \Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)\cdot p^{<k+1>}_{j_1\ldots j_n}(\tau,z)=$$

$$=\sum_{j_1,\ldots,j_n\in\mathbf{Z}} \Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)\cdot p^{<k+1>}_{j_1\ldots j_n}(\tau,z)=q^{<k+1>}_{i_1\ldots i_n}(\tau,\tau_0,z) \qquad (1.15)$$

because of the following lemma.

Lemma 1.1.

$$\frac{\partial\Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)}{\partial t_0}=-\sum_{\alpha_1,\ldots,\alpha_n\in\mathbf{Z}} \Phi_{i_1\ldots i_n\alpha_1\ldots\alpha_n}(\tau_0,\tau)\cdot h_{\alpha_1\ldots\alpha_n j_1\ldots j_n}(\tau)$$

Proof of Lemma 1.1. We start from the following identity

$$\sum_{j_1,\ldots,j_n\in\mathbf{Z}} \Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)\cdot\Phi_{j_1\ldots j_n\alpha_1\ldots\alpha_n}(\tau,\tau_0)=\delta_{i_1\alpha_1}\cdots\delta_{i_n\alpha_n} \qquad (1.16)$$

which is a consequence of (1.11) and (1.12). By differentiation of the above identity in respect to τ, we obtain

$$\sum_{j_1,\ldots,j_n\in\mathbf{Z}} \frac{\partial\Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)}{\partial t_0}\cdot\Phi_{j_1\ldots j_n\alpha_1\ldots\alpha_n}(\tau,\tau_0)+$$

$$+\sum_{j_1,\ldots,j_n\in\mathbf{Z}} \Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)\cdot\frac{\partial\Phi_{j_1\ldots j_n\alpha_1\ldots\alpha_n}(\tau,\tau_0)}{\partial\tau}=0$$

i.e. having in mind the identity (1.12),

$$\sum_{j_1,\ldots,j_n\in\mathbf{Z}} \frac{\partial\Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)}{\partial t_0}\cdot\Phi_{j_1\ldots j_n\alpha_1\ldots\alpha_n}(\tau,\tau_0)=$$

$$=-\sum_{j_1,\ldots,j_n\in\mathbf{Z}} \Phi_{j_1\ldots j_n\alpha_1\ldots\alpha_n}(\tau,\tau_0)\sum_{\beta_1,\ldots,\beta_n\in\mathbf{Z}} \Phi_{i_1\ldots i_n\beta_1\ldots\beta_n}(\tau_0,\tau)\cdot h_{\beta_1\ldots\beta_n j_1\ldots j_n}(\tau)$$

The proof of Lemma 1.1 is completed by the following lemma.

Lemma 1.2. *If*

$$\sum_{j_1,\ldots,j_n\in\mathbf{Z}} a_{j_1\ldots j_n}(\tau,\tau_0)\cdot\Phi_{j_1\ldots j_n\alpha_1\ldots\alpha_n}(\tau,\tau_0)=0\quad\forall\alpha_1,\ldots,\alpha_n\in\mathbf{Z}\tag{1.17}$$

for τ in a neighborhood of τ_0, then

$$\forall j_1,\ldots,j_n\in\mathbf{Z}\quad a_{j_1\ldots j_n}(\tau,\tau_0)=0\tag{1.18}$$

Proof. Multiplying the identity (1.17) with $\Phi_{\alpha_1\ldots\alpha_n i_1\ldots i_n}(\tau_0,\tau)$ and summing up in respect to $\alpha_1,\ldots,\alpha_n\in\mathbf{Z}$, we obtain

$$\sum_{\alpha_1,\ldots,\alpha_n\in\mathbf{Z}}\Phi_{\alpha_1\ldots\alpha_n i_1\ldots i_n}(\tau_0,\tau)\sum_{j_1,\ldots,j_n\in\mathbf{Z}} a_{j_1\ldots j_n}(\tau,\tau_0)\cdot\Phi_{j_1\ldots j_n\alpha_1\ldots\alpha_n}(\tau,\tau_0)=0$$

i.e.

$$\sum_{j_1,\ldots,j_n\in\mathbf{Z}} a_{j_1\ldots j_n}(\tau,\tau_0)\sum_{\alpha_1,\ldots,\alpha_n\in\mathbf{Z}}\Phi_{j_1\ldots j_n\alpha_1\ldots\alpha_n}(\tau,\tau_0)\cdot\Phi_{\alpha_1\ldots\alpha_n i_1\ldots i_n}(\tau_0,\tau)=0$$

i.e., using (1.16),

$$\sum_{j_1,\ldots,j_n\in\mathbf{Z}} a_{j_1\ldots j_n}(\tau,\tau_0)\cdot\delta_{j_1 i_1}\cdots\delta_{j_n i_n}=a_{i_1\ldots i_n}(\tau,\tau_0)=0\quad\blacksquare$$

Using (1.15), we obtain

$$q^{<k+1>}_{i_1\ldots i_n}(t,t_0,z)=\frac{\partial}{\partial t}q^{<k>}_{i_1\ldots i_n}(t,t_0,z)$$

and, by induction,

$$q^{<k>}_{i_1\ldots i_n}(t,t_0,z)=\frac{\partial^k}{\partial t^k}q^{<0>}_{i_1\ldots i_n}(t,t_0,z)\ ,\quad k=0,1,\ldots\tag{1.19}$$

We shall prove the following lemma.

Lemma 1.3.

$$\left|p^{<k>}_{\alpha_1\ldots\alpha_n}(\tau,z)\right|\le\frac{k!}{\rho^k}\sum_{i_1,\ldots,i_n\in\mathbf{Z}}|\Phi_{\alpha_1\ldots\alpha_n i_1\ldots i_n}(\tau,\tau_0)|\cdot q_{i_1\ldots i_n}(\tau,\tau_0,z)\tag{1.20}$$

where

$$q_{i_1\ldots i_n}(t,t_0,z)=\sup_{t\in D_t}\left|q^{<0>}_{i_1\ldots i_n}(t,t_0,z)\right|\ ,\quad D_t=\{\zeta\in\mathbf{C}:\ |\zeta-t|=\rho\}\tag{1.21}$$

Proof. The identity (1.14) is invertible. Namely, we shall prove that

$$p^{<k>}_{\alpha_1\ldots\alpha_n}(\tau,z)=\sum_{i_1,\ldots,i_n\in\mathbf{Z}}\Phi_{\alpha_1\ldots\alpha_n i_1\ldots i_n}(\tau,\tau_0)\cdot q^{<k>}_{i_1\ldots i_n}(\tau,\tau_0,z)$$

Multiplying the identity (1.14) with $\Phi_{\alpha_1\ldots\alpha_n i_1\ldots i_n}(\tau,\tau_0)$ and summing up in respect to $i_1,\ldots,i_n\in\mathbf{Z}$, we obtain

$$\sum_{i_1,\ldots,i_n\in\mathbf{Z}} \Phi_{\alpha_1\ldots\alpha_n i_1\ldots i_n}(\tau,\tau_0)\cdot q^{<k>}_{i_1\ldots i_n}(\tau,\tau_0,z) =$$

$$= \sum_{i_1,\ldots,i_n\in\mathbf{Z}} \Phi_{\alpha_1\ldots\alpha_n i_1\ldots i_n}(\tau,\tau_0) \sum_{j_1,\ldots,j_n\in\mathbf{Z}} \Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau)\cdot p^{<k>}_{j_1\ldots j_n}(\tau,z) =$$

$$= \sum_{j_1,\ldots,j_n\in\mathbf{Z}} p^{<k>}_{j_1\ldots j_n}(\tau,z) \sum_{i_1,\ldots,i_n\in\mathbf{Z}} \Phi_{\alpha_1\ldots\alpha_n i_1\ldots i_n}(\tau,\tau_0)\cdot \Phi_{i_1\ldots i_n j_1\ldots j_n}(\tau_0,\tau) =$$

$$= \sum_{j_1,\ldots,j_n\in\mathbf{Z}} p^{<k>}_{j_1\ldots j_n}(\tau,z)\delta_{\alpha_1 j_1}\cdots\delta_{\alpha_n j_n} = p^{<k>}_{\alpha_1\ldots\alpha_n}(\tau,z)$$

according to (1.16).

By (1.19) and the Cauchy integral formula [2], we have

$$\left|q^{<k>}_{i_1\ldots i_n}(t,t_0,z)\right| \le \frac{k!}{\rho^k}\cdot q_{i_1\ldots i_n}(t,t_0,z)$$

where $q_{i_1\ldots i_n}(z)$ is given by (1.21). Further we have

$$\left|p^{<k>}_{\alpha_1\ldots\alpha_n}(\tau,z)\right| = \left|\sum_{i_1,\ldots,i_n\in\mathbf{Z}} \Phi_{\alpha_1\ldots\alpha_n i_1\ldots i_n}(\tau,\tau_0)\cdot q^{<k>}_{i_1\ldots i_n}(\tau,\tau_0,z)\right| \le$$

$$\le \sum_{i_1,\ldots,i_n\in\mathbf{Z}} \left|\Phi_{\alpha_1\ldots\alpha_n i_1\ldots i_n}(\tau,\tau_0)\right|\cdot\left|q^{<k>}_{i_1\ldots i_n}(\tau,\tau_0,z)\right| \le$$

$$\le \sum_{i_1,\ldots,i_n\in\mathbf{Z}} \left|\Phi_{\alpha_1\ldots\alpha_n i_1\ldots i_n}(\tau,\tau_0)\right|\cdot\frac{k!}{\rho^k}\cdot q_{i_1\ldots i_n}(\tau,\tau_0,z) \quad \blacksquare$$

Having in mind Lemma 1.3, we have

$$|x_{i_1\ldots i_n}| = \left|\sum_{k=0}^{\infty}\frac{(t_0-t)^k}{k!} p^{<k>}_{i_1\ldots i_n}\right| \le$$

$$\le \sum_{k=0}^{\infty}\frac{|t_0-t|^k}{\rho^k} \sum_{j_1,\ldots,j_n\in\mathbf{Z}} \left|\Phi_{i_1\ldots i_n j_1\ldots j_n}(t,t_0)\right| q_{j_1\ldots j_n}(t,t_0,z)$$

Hence we have absolute and uniform convergence in (1.10) for t in a neighborhood of t_0. According to the Weierstrass theorem [2], the series $x_{i_1\ldots i_n}$ is a regular function in a neighborhood of t_0 and it is allowed to be differentiated by terms. By differentiation, in [3] we obtain the result (1.9). ∎

Remark. Unlike the proof in [3], which is mostly theoretical, the above proof can be used for analysis of the error made by taking a finite number of terms in (1.9).

Conclusions

In this paper a functional series solution of the system of nonlinear ODE (1.3), is presented. A further research could be to specify this series to a system with a control variable explicitly appearing in the system equations. Also, in this paper, an alternative proof of Theorem 1.1 is presented, besides the proof in [3], which contains a closer estimation of the error made by taking a finite number of terms in the functional series solution. A further research is oriented on developing a numerical algorithm for solving systems of ODE, based on the paper result.

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