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On the relation between some problems in Number theory, Orthogonal polynomials and Differential equations

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Abstract: The author discuss a special procedure and deduces two theorems, which proofs are based on the Analysis and some properties of Chebishev polynomials.

Key words: *monotonic functions, Chebishev polynomials*, *divisor problem* in number theory, and *differential equations* of second order.

1. Introduction

Let $\tau(n)$ be the number of positive divisors of a natural integer number n and let $T(n)$ be its summatory function. Then we have

$$T(n) = \sum_{m \le n} \tau(m) = \sum_{m \le n} \sum_{xy=m} 1 = \sum_{x \le n} \left[\frac{n}{x} \right],$$

where $[a]$ means the integer parts of a.

Firstly P. G. Lejene Dirichlet [1] in 1849 evaluated $T(n)$, proving that

(1) $$T(n) = n(\ln n + 2\gamma - 1) + \Delta(n)$$

and

(2) $$\Delta(n) = O(\sqrt{n}),$$

where $\gamma = 0{,}5772...$ is the Euler constant.

Dirichlet avoided the trivial evaluation

$$T(n)=\sum_{x\le n}\frac{n}{x}+O(n)=n\ln n+O(n)$$

by using the identity

(3) $$T(n)=-\mu\nu+\sum_{x\le\mu}\left[\frac{n}{x}\right]+\sum_{x\le\nu}\left[\frac{n}{\nu}\right],$$

where $\mu(\mu+1)\ge n,\ \nu=\left[\frac{n}{\mu}\right]$.

From (3) for $\mu=\nu=\lfloor\sqrt{n}\rfloor$ it follows

(4) $$T(n)=2\sum_{x\le\sqrt{n}}\left[\frac{n}{x}\right]-\left[\sqrt{n}\right]^2.$$

Dirichlet deduced (1) and (2) directly from (4) by using the equality

$$\sum_{x\le\sqrt{n}}\left[\frac{n}{x}\right]=\sum_{x\le\sqrt{n}}\frac{n}{x}-\rho_n\left(\sqrt{n}\right)=n\left(\ln\sqrt{n}+\gamma\right)+O\left(\frac{1}{\sqrt{n}}\right)-\rho_n\left(\sqrt{n}\right),$$

where

$$\rho_n(m)=\sum_{x\le m}\left(\frac{n}{x}-\left[\frac{n}{x}\right]\right),$$

and $\rho_n(m)=O(m)$ as a trivial estimation.

The so called "Divisor problem" (of Dirichlet) consists of estimating the error term $\Delta(n)$ in (1) with possibly better order. The brief history consists of the following records:

Voronoj [2] in 1903: $O\left(n^{\frac{1}{3}}\ln n\right)$, by using Farrey sequences;

Van der Corput [3] in 1928: $O\left(n^{\frac{27}{82}}\right)$, with special exploration of exponential sums;

Kolesnik [4] in 1985: $O\left(n^{\frac{139}{429}}\right)$, who approved the method of Van der Corput;

Iwaniec and Mozzochi [5] in 1988: $O\left(n^{\frac{7}{22}}\right)$, who studied the number of integer points not only beneath the hyperbola (Divisor problem), but also the number of integer points in the circle (Circle problem) with equal success.

The last record is due to M. Huxley [6], who proved that $\Delta(n) = O\left(n^{\frac{23}{73}}(\ln n)^{\frac{461}{146}}\right)$. Huxley studied the number of integer points (lattice points) in a closed curve, as well as the number of unit squares which centres lie within the curve.

The lattice theory or *gitterpunktlehre* origins from Gauss`s investigation about the integer points in the circle and from the cited above Dirichlet`s work [1].

2. About a special procedure

Now we will describe a special procedure, which we described firstly in [7] in 1994. We generalize the idea of Dirichlet, hidden in the identity (3) (in (4) particularly). Geometrical point of view shows, that Dirichlet divided the graph of the hyperbola $y = n/x$ into two pieces by the dividing point $x = \mu$ (particularly $x = [\sqrt{n}]$) and in the second piece the roles of x and y are changed.

We divide the graph of the line $y = f(x)$ into pieces by the following procedure.

Let $y = f(x)$ be a real monotonic (strictly increasing or decreasing) function in the interval $[a, b]$ and let $\varphi(x)$ be its inverse function. We divide the graph of $f(x)$ into $k+1$ parts by dividing points x_1, $x_2, \ldots$, x_κ, $a < x_1 < x_2 < \ldots x_k < b$ by the following way: the first, the third etc parts (for the intervals (a, x_1), (x_2, x_3) etc.) we project orthogonally on the axe Ox; the second, the fourth etc. parts (for the intervals (x_1, x_2), (x_3, x_4) etc.) we project orthogonally on Oy; all projections must be equal, and tend to 0, when $\kappa \to \infty$.

This procedure gives us the following equations, if $f(x)$ is increasing:

(5`) $$x_1 - a = x_3 - x_2 = \ldots = \begin{cases} x_{2r-1} - x_{2r-2}, & k = 2r-1; \\ x_{2r-1} - x_{2r-2} = b - x_{2r}, & k = 2r; \end{cases} =$$

(5``) $$= f(x_2) - f(x_1) = \ldots = f(x_{2r-2}) - f(x_{2r-3}) =$$
$$= \begin{cases} f(b) - f(x_{2r-1}), & k = 2r-1; \\ f(x_{2r}) - f(x_{2r-1}), & k = 2r. \end{cases}$$

We will eliminate x_2, $x_3, \ldots,$ x_κ. From (5`) and (5``) we get

$$f(x_2) = x_1 - a + f(x_1);, \ x_2 = \varphi(x_1 - a + f(x_1));$$
$$x_3 = x_1 - a + x_2 = x_1 - a + \varphi(x_1 - a + f(x_1));$$
$$f(x_4) = x_1 - a + f(x_3); \ x_4 = \varphi(x_1 - a + f(x_1 - a + \varphi(x_1 - a + f(x_1))));$$
$$x_5 = x_1 - a + x_4 = x_1 - a + \varphi(x_1 - a + f(x_1 - a + \varphi(x_1 - a + f(x_1))));$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f(x_{2s}) = x_1 - a + f(x_{2s-1}); \ x_{2s} = \varphi(x_1 - a + f(x_1 - a + \ldots + f(x_1)\ldots));$$
$$x_{2s+1} = x_1 - a + x_{2s} = x_1 - a + \varphi(x_1 - a + f(x_1 - a + \varphi(x_1 - a + \ldots$$
$$+ \varphi(x_1 - a + f(x_1))\ldots)))$$

where we have s times φ and f, $1 \le s \le \dfrac{k-1}{2}$.

But

$f(b) - f(x_{2r-1}) = x_1 - a$ and $b = \varphi(x_1 - a + f(x_{2r-1}))$ for $k = 2r-1$;

$b - x_{2r} = x_1 - a$ and $b = x_1 - a + x_{2r} = x_1 - a + \varphi(x_1 - a + f(x_{2r-1}))$ for $k = 2r$,

and we have

$$b = F_{2r}(z_r) \text{ for } k = 2r-1,$$

and

$$b = z_r + F_{2r}(z_r) \text{ for } k = 2r,$$

where $z_r = x_1 - a$ and

(6) $$F_{2r}(z) = \varphi\big(z + f\big(z + \varphi\big(z + f\big(z + \ldots + \varphi\big(z + f(z+a)\big)\ldots\big)\big)\big)\big)$$

(r times φ and f).

We get similar results, if $f(x)$ is decreasing (in the system of equations we will have the differences $f(x_i) - f(x_{i+1})$ instead of $f(x_{i+1}) - f(x_i)$ for the case of increasing functions):

$$b = G_{2r}(z_r) \text{ for } k = 2r-1$$

and

$$b = z_r + G_{2r}(z_r) \text{ for } k = 2r,$$

where $z_r = x_1 - a$ and

(7) $$G_{2r}(z) = \varphi\big(-z + f\big(z + \varphi\big(-z + f\ \big(z + \ldots + \varphi\big(-z + f(z+a)\big)\ldots\big)\big)\big)\big)$$

(r times φ and f).

The sum of all projections will be: $L_k = (k+1)z_k$. If $k \to \infty$, then $x_1 \to a$ and $z_r \to 0$. We introduce also the designation $L = \lim_{k\to\infty}(k+1)z_r$. This constant, when exists, characterizes very special properties of $f(x)$, but we will not discuss L here.

3. The relation between orthogonal polynomials and differential equations

The relation between orthogonal polynomials and differential equations is well known. This relation is presented by the equations of the form

$$A(x)\ y'' + \big(B(x) + A'(x)\big)\, y' + C\ y = 0,$$

where $A(x)$ and $B(x)$ are polynomials, and C is a constant.

In the particular case of the equation

$$(1 - x^2)\ y'' - x\ y' + n^2\ y = 0,$$

its roots are the Chebyshev`s polynomials of first kind. On the other hand, the roots of these polynomials can be find geometricaly: we divide the semicircumference into parts with equal length; then the projections of the dividing points on Ox are the roots (see for instance [8], p. 69). This procedure is different from our. But we will show that the Chebyshev`s polynomials are in liaison with our procedure too, proving the following

Theorem 1. If $f(x) = \dfrac{1}{x}$ in the interval (α, β), $0 < \alpha < \beta$, then

(8) $$G_{2s}(z) = -\frac{\sin\left(2s \arccos\dfrac{z}{2}\right) + \alpha \sin\left((2s-1)\arccos\dfrac{z}{2}\right)}{\sin\left((2s+1)\arccos\dfrac{z}{2}\right) + \alpha \sin\left(2s \arccos\dfrac{z}{2}\right)},$$

where $G_{2s}(z)$ is defined in (7).

Proof. Let $f(x)=\frac{1}{x}$ in the interval (α,β), $\left(\alpha=\frac{a}{\sqrt{n}},\ \beta=\frac{b}{\sqrt{n}}\right)$. Then $\varphi(x)=f(x)=\frac{1}{x}$. We put

(9) $$G_0(z)=G_0=\alpha.$$

Then from (7) we get

(10) $$G_2(z)=\varphi(-z+f(z+G_0))=\frac{1}{-z+\frac{1}{z+G_0}}=\frac{z+G_0}{-z^2-G_0z+1}=$$
$$=-\frac{z+G_0}{z^2+G_0z-1},$$

and in general,

(11) $$G_{2s}(z)=\varphi(-z+f(z+G_{2s-2}(z)))=\frac{1}{-z+\frac{1}{z+G_{2s-2}(z)}}=.$$
$$=-\frac{z+G_{2s-2}(z)}{z^2-1+zG_{2s-2}(z)}.$$

Let us introduce the following system of polynomials:

(12) $$H_0=1,\ H_1=z,$$

and for any integer $s\geq 1$

(13) $$H_{s+1}(z)=zH_s(z)-H_{s-1}(z)$$

It is easy to prove that

(14) $$G_0(z)=\alpha H_0,$$

and

(15) $$G_{2s}(z)=-\frac{H_{2s-1}(z)+\alpha H_{2s-2}(z)}{H_{2s}(z)+\alpha H_{2s-1}(z)}.$$

The equality (14) follows from (9) and (12). The equality (15) can be proved by mathematical induction. Indeed, for $s=1$, as $H_2(z)=z^2-1$, we have

$$G_2(z)=-\frac{H_1(z)+\alpha H_0(z)}{H_2(z)+\alpha H_1(z)}=-\frac{z+\alpha}{z^2-1+\alpha z},$$

which coincides with (11) for $s=1$.

Let be (15) for some natural s. Then for $s+1$ we will have

$$G_{2s+2}(z) = -\frac{z+G_{2s}(z)}{z^2-1+zG_{2s}(z)} = -\frac{z - \dfrac{H_{2s-1}+\alpha H_{2s-2}}{H_{2s}+\alpha H_{2s-1}}}{z^2-1-\dfrac{zH_{2s-1}-\alpha zH_{2s-2}}{H_{2s}+\alpha H_{2s-1}}} =$$

$$= -\frac{zH_{2s}\alpha zH_{2s-1}-H_{2s-1}-\alpha H_{2s-2}}{(z^2-1)(H_{2s}+\alpha H_{2s-1})-zH_{2s-1}+\alpha zH_{2s-2}} =$$

$$= -\frac{zH_{2s}-H_{2s-1}+\alpha z(zH_{2s-1}-H_{2s-2})}{z(zH_{2s}-H_{2s-1})-H_{2s}+\alpha z(zH_{2s-1}-H_{2s-2})-\alpha H_{2s-1}} =$$

$$= -\frac{H_{2s+1}+H_{2s}}{zH_{2s+1}-H_{2s}+\alpha(zH_{2s}-H_{2s-1})} = -\frac{H_{2s+1}+\alpha H_{2s-1}}{H_{2s+2}+\alpha H_{2s+1}}$$

or

$$G_{2s+2}(z) = -\frac{H_{2s+1}+\alpha H_{2s-1}}{H_{2s+2}+\alpha H_{2s+1}},$$

which is the equality (15) for $s+1$. So (15) is proved by mathematical induction.

But the linear relation (13) shows, that H_0, H_1, H_2, … are orthogonal polynomials as a variant of the well known polynomials of Chebyshev of second kind $\widetilde{U}_n(x)$, defined by the following relations:

$$\widetilde{U}_0(x) = 1,\ \widetilde{U}_1(x) = x, \text{ and}$$

(16) $$\widetilde{U}_{n+1}(x) = x\widetilde{U}_n(x) - \frac{1}{4}\widetilde{U}_{n-1}(x) \text{ for } n \ge 1.$$

We will prove, that for the polynomials $H_n(z)$ we have

(17) $$H_n(z) = 2^n \widetilde{U}_n\left(\frac{z}{2}\right).$$

Really, $H_0(z) = 2^0\widetilde{U}_0\left(\frac{z}{2}\right) = 1$, $H_1(z) = 2\,\widetilde{U}_1\left(\frac{z}{2}\right) = 2.\frac{z}{2} = z$, and the relation (13), by the hypotheses (17), became

$$2^{n+1}\widetilde{U}_{n+1}\left(\frac{z}{2}\right) = z2^n\widetilde{U}_n\left(\frac{z}{2}\right) - 2^{n-1}\widetilde{U}_{n-1}\left(\frac{z}{2}\right),$$

or

$$\widetilde{U}_{n+1}\left(\frac{z}{2}\right) = \frac{z}{2}\widetilde{U}_n\left(\frac{z}{2}\right) - \frac{1}{4}\widetilde{U}_{n-1}\left(\frac{z}{2}\right),$$

which is (16) for $x = \frac{z}{2}$.

The polynomials of Chebyshev $\widetilde{U}_n(x)$ are orthogonal in $(-1,1)$ with respect to $\sqrt{1-x^2}$. This means, that the polynomials $H_n(z)$ are orthogonal in $(-2,2)$ with respect to $\sqrt{4-x^2}$.

The very known explicite expression of $\widetilde{U}_n(x)$ is

$$\widetilde{U}_n(x) = \frac{1}{2^n}\frac{\sin\left((n+1)\arccos x\right)}{\sqrt{1-x^2}}$$

and we have

$$H_n(z) = 2^n \widetilde{U}_n\left(\frac{z}{2}\right) = \frac{2\sin\left((n+1)\arccos\frac{z}{2}\right)}{\sqrt{4-z^2}}. \tag{18}$$

Substituting (18) in (15), we receive (8) and the theorem 1 is proved.

Remark. For some calculations equality (15) can by more suitable, and we can formulate as proved the following

Theorem 2. If $f(x) = \frac{1}{x}$ in the interval (α, β), $0 < \alpha < \beta$, then

$$G_{2s}(z) = -\frac{H_{2s-1}(z) + \alpha H_{2s-2}(z)}{H_{2s}(z) + \alpha H_{2s-1}(z)}$$

with the previous notations.

For clarity of our procedure, we present the following

An Example. For the case of hyperbola $f(x) = \varphi(x) = \frac{1}{x}$, $x \in (\alpha, \beta)$, $0 < \alpha < \beta$ with 4 dividing points, $k = 2r$, $r = 2$, we will have the following system:

$$x_1 - \alpha = x_3 - x_2 = \beta - x_4 = f(x_1) - f(x_2) = f(x_3) - f(x_4),$$

which is equivalent to

$$\begin{cases} x_2 = \varphi\left(-(x_1-\alpha) + f(x_1)\right) \\ x_3 = x_1 - \alpha + \varphi\left(-(x_1-\alpha) + f(x_1)\right) \\ x_4 = \varphi\left(-(x_1-\alpha) + f(x_3)\right) = \varphi\left(-(x_1-\alpha) + f(x_1 - \alpha + \varphi\left(-(x_1 = \alpha) + f(x_1)\right))\right) \\ x_1 - \alpha = \beta - x_4 \end{cases}$$

or

$$(19)\qquad \begin{cases} x_2 = \dfrac{1}{-z+\dfrac{1}{z+\alpha}} \\ x_3 = z+\dfrac{1}{-z+\dfrac{1}{z+\alpha}} \\ x_4 = \dfrac{1}{-z+\dfrac{1}{z+\dfrac{1}{-z+\dfrac{1}{-z+\dfrac{1}{z+\alpha}}}}}, \\ z = x_1-\alpha = \beta - x_4 \end{cases}$$

where z is the same as $z_r = z_2$ in the previous notation.

For the function $G_{2r}(z)$ we will have

$$G_4(z) = x_4 = \varphi\big(-(x_1-\alpha)+f(x_3)\big) = \varphi\Big(-(x_1-\alpha)+f\big(x_1-\alpha+\varphi\big(-(x_1-\alpha)+f(x_1)\big)\big)\Big) =$$
$$= \varphi\Big(-z+f\big(z+\varphi\big(-z+f(z+\alpha)\big)\big)\Big)$$

or

$$G_4(z) = \frac{1}{-z+\dfrac{1}{z+\dfrac{1}{-z+\dfrac{1}{z+\alpha}}}} = \frac{1}{-z+\dfrac{1}{z+\dfrac{z+\alpha}{-z^2-z\alpha+1}}} =$$
$$= \frac{1}{-z+\dfrac{-z^2-z\alpha+1}{-z^3-z^2\alpha+2z+\alpha}} =$$

$$= \frac{-z^3-z^2\alpha+2z+\alpha}{z^4+z^3\alpha-2z^2-\alpha z-z^2-z\alpha+1} = -\frac{z^3-2z+\alpha\left(z^2-1\right)}{z^4-3z^2+1+\alpha\left(z^3-2z\right)} =$$

.

$$= -\frac{H_3(z)+\alpha H_2(z)}{H_4(z)+\alpha H_3(z)}$$

It is easily to calculate the system (19) by using the computer system Mathematica, (where ChebyshevU[n, $z/2$] : $=H_n(z)$). We receive, with 5-digit precision,

Solve[{x2==1/(-z+1/(z+o.25)),

x3 == z+1/(-z+1/(z+0.25

x4 == 1/(-z+1/(z+1/(-z+1/(z+0.25)))), z == 4-x4

x1 == 0.25+ 4-x4} , {x1, x2, x3, x4, z}]

{{x1→ -1.43153, x2 →1.01731,

x3 →-0.66422, x4 →5.68153, z →-1.68153},

{x1 →-0.538593, x2 →-0.936247, x3 →-1.72484,

x4 →4.78859, z →-0.788593}, {x1 →0.68284,

x2 →0.969339, x3 →1.40218, x4 →3.56716, z →0.43284},

{x1 →1.78729, x2 →-1.02272, x3 →0.514575, x4 →2.46271,

z →1.53729}, {x1 →4.49999, x2 →-0.248276,

x3 →4.00172, x4 →-0.249994, z →4.24999}}

For our geometrical problem the only solution is: $x_1 = 0{,}68284$, $x_2 = 0{,}969339$, $x_3 = 1{,}40218$, $x_4 = 3{,}56716$. We illustrate our procedure in fig. 1. We comparing it with one dividing point x_1 –the method of Dirichlet – as the positive solution of the equation $x_1 - \alpha = \frac{1}{x_1} - \frac{1}{\beta}$ ($\alpha = 0{,}25$, $\beta = 4$), from where $x_1 = \frac{\alpha\beta - 1 + \sqrt{4b^2 + (\alpha\beta - 1)^2}}{2\beta} = 1$ (fig.2).

In the case of 4 dividing points we have $L_4 = 5(x_1 - \alpha) = 5(0{,}68284 - 0{,}25) = 2{,}1642$; in the case of one dividing point we have $L_1 = 2(x_1 - \alpha) = 2(1 - 0{,}25) = 1{,}5$ and $L_4 > L_1$. But if $\alpha = 0{,}5$, $\beta = 1{,}5$, then $L_4 = 1{,}01888$, $L_1 = 1{,}840266$ and $L_4 < L_1$.

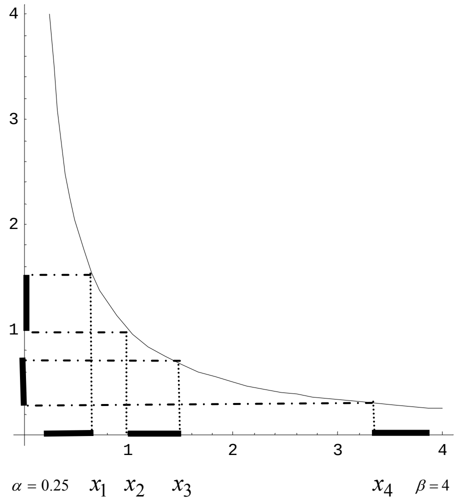

Fig. 1

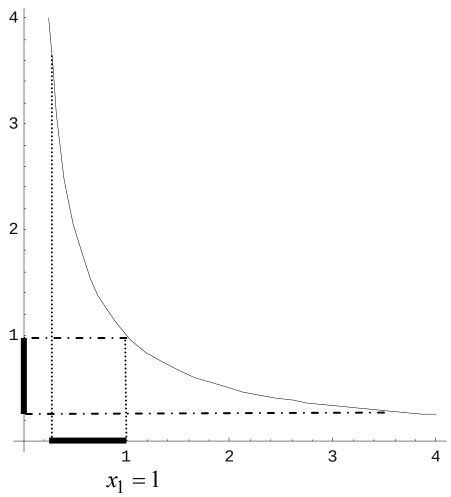

Fig. 2

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