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ONE PROOF FOR A THEOREM OF DIFFERENTIAL EQUATIONS

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Abstract
In this article we present a proof for a known theorem of differential equations conserning to the boundary problem.The proof is given by means of distributions. Similar proof is given (2.p.275)

Introduction
The object of consideration is the differential equation of the form

$$a_n y^{(n)} + ... + a_0 y = u \tag{1}$$

where $a_n,...,a_0$, are complex number, $a_n \neq 0$, y is the unknown function and u is a given function.

By $L = a_n \frac{d^n}{dt^n} + ... + a_1 \frac{d}{dt} + a_o$ one denote the differential operator of order n . This means $Ly = a_n y^{(n)} + ... + a_0 y,$ so that we can write

$$Ly = u \tag{2}$$

If on right side a given distribution U ; then Y is the unknown distribution of $D^{'}, D^{'}$ is the space of Schwartz distribution .
In this case one write

$$\text{LY=U} \tag{3}$$

In the theory of differential equations the basic meaning has the following distribution $L\delta = \sum_{k=0}^{n} a_k \delta^{(k)}$, δ is the Dirac distribution.

Further by * one denote the operation convolution of distribution : if $T,U \in D^{'}$ then T*U is there convolution. In general the convolution is not defined for every two distribution. Moreover if at least one has compact support then the convolution is well defined and holds the relations

$$T * U = 2\pi F(F^{-1}T \cdot F^{-1}U) = F^{-1}(FT \cdot FU)$$

where F denote the Fourier transform of distribution.
If is also well known that $\delta^{(n)} * T = T^{(n)}$ from this the equation (3) can be written in the form

$$L\delta * Y = U \tag{4}$$

From the previous facts we have

$$L\delta * Y = 2\pi F\left(F^{-1}L\delta \cdot F^{-1}Y\right)$$

Since

$$F^{-1}L\delta = 2\pi\left[a_n(it)^n + \ldots + a_1(it) + a_o\right] = Q_n(t),$$
$$Q_n(t) = 2\pi\left[a_n(it)^n + \ldots + a_o\right] = A(t-t_1)^{k_1}\ldots(t-t_m)^{k_m}$$

where $A = 2\pi a_n i^n$, $t_1,\ldots,t_m$ are the roots of the polinomail $Q_n(t)$ and $k_1,\ldots,k_m$ are the multiplicity . Thus on get

$$2\pi F\left(Q_n(t)F^{-1}Y\right) = U$$

other

$$2\pi Q_n(t)F^{-1}Y = F^{-1}U, F^{-1}Y = \frac{1}{2\pi}\frac{F^{-1}U}{Q_n(t)}, Y = \frac{1}{2\pi}F\left(\frac{F^{-1}U}{Q_n(t)}\right)$$

Consequntly the solution is given by $Y = \frac{1}{2\pi}F\left(\frac{F^{-1}U}{Q_n(t)}\right)$ if the Fourier transforms exists for the distributions.
Here we consider the differential equation

$$a_n y^{(n)} + \ldots + a_o y = u \tag{5}$$

u is a continous function on the interval $[t_o, \infty]$ and y is the unknown function which must satisfy (5) on the interval $[t_0, \infty]$. It is to find the function $y(t)$ which satisfy (5) and the boundary conditions $y^{(\nu)}(t_0) = y_\nu$, $\nu = 0,1,...,n-1$.

Here we will give one proof that such a function exist on the interval $\Omega = (t_0, \infty)$, $y \in C^n(\Omega)$ and $\lim_{t \to t_0} y^{(k)}(t) = y_k$. For a similar proof see for example (2.p275)

Before we pass to the proof we will consider the distribution $\Gamma = gH, H$ is the Heaviside function and g is the solution for the differential equations , $g(0) = g^{'}(0) = ... = g^{(n-2)}(0) = 0$ and $g^{(n-1)}(0) = \frac{1}{a_n}$. The distribution gH is the inverse for the distribution $L\delta = \sum_{k=0}^{n} a_k \delta^{(k)}$; this means $L\delta * gH = \delta$. By $D_+^{'}$ we denote the distributions which has support in the intervals of the form $[t_1, \infty]$ $t_1 \in R$.

It is of interes to remark that $D_+^{'}$ with the operation convolution of distributions is a commutative algebra with δ the identitiy element , and without multipliers of zero.

Now we pass to the proof of theorem.

Proof. By assuming that there exist a function $y(t)$ of class $C^n(t_0, \infty)$ which satisfy the conditions $\lim_{t \to t_0} y^{(k)}(t) = y_k, k = 0,1,...n-1;$ we consider the following distributions $y(t)H(t-t_0)$ and $u(t)H(t-t_0)$ of the algebra $D_+^{'}$. From the hypothesis it easy to prove that $[y(t)H(t-t_0)]^{(k)} = y^{(k)}(t)H(t-t_0) + y_{k-1}\delta(t-t_0) + ... + y_0\delta^{(k-1)}(t-t_0)$ in distributional sense for k=1,2,…,n.

With this in account on obtain

$$L[y(t)H(t-t_0)] = (Ly)H(t-t_0) + \sum_{k=0}^{n-1} b_k \delta^{(k)}(t-t_0) \qquad (6)$$

$$b_k = \sum_{j=0}^{n-k-1} a_{k+j+1} y_j$$

Since $(Ly)H(t-t_0) = u(t)H(t-t_0)$ we have

$$L\delta * [y(t)H(t-t_0)] = u(t)H(t-t_0) + \sum_{k=0}^{n-1} b_k \delta^{(k)}(t-t_0) \qquad (7)$$

After multiplication the equation (7) with distribution gH on get

$$L\delta * gH * [yH(t-t_0)] = u(t)H(t-t_0) * g(t)H(t) + \sum_{k=1}^{n-1} b_k \delta^{(k)}(t-t_0) * gH$$

Because $L\delta * gH = \delta, \delta^{(k)}(t-t_0) * gH = (gH)^{(k)}(t-t_0) = g^{(k)}(t-t_0)$ for k=0,1,…,n-1 ; definitely on obtain

$$y(t)H(t-t_0) = \int_{t_0}^{t} u(\tau)g(t-\tau)d\tau + \sum_{k=0}^{n-1} b_k g^{(k)}(t-t\ ), t > t_0 \qquad (8)$$

$$y(t) = \int_{t_0}^{t} u(\tau)g(t-\tau)d\tau + \sum_{k=0}^{n-1} b_k g^{(k)}(t-t_0), t > t_0 \qquad (9)$$

The following formulas holds for the function (9) :

$$y^{(j)}(t) = \int_{t_0}^{t} u(\tau)g^{(j)}(t-\tau)d\tau + g^{(j-1)}(0)u(t) + \sum_{k=0}^{n-1} b_k g^{(k+j)}(t-t_0), \qquad (10)$$

$j = 0,1,...,n-1$

Thus

$$y^{(j)}(t) = \int_{t_0}^{t} u(\tau)g^{(j)}(t-\tau)d\tau + \sum_{k=0}^{n-1} b_k g^{(k+j)}(t-t_0), \ j = 0,1,...,n-1 \qquad (11)$$

and

$$y^{(n)}(t) = \int_{t_0}^{t} u(\tau)g^{(n)}(t-\tau)d\tau + \frac{1}{a_n}u(t) + \sum_{k=0}^{n-1} b_k g^{(k+n)}(t-t_0) \qquad (12)$$

If , for example , $u(t) \in C^1[t_0, \infty)$ on get

$$y^{(n+1)}(t) = \int_{t_0}^{t} u(\tau)g^{(n+1)}(t-\tau)d\tau + g^{(n)}(0)u(t) + \frac{1}{a_n}u'(t) + \sum_{k=0}^{n-1} b_k g^{(k+n+1)}(t-t_0)$$

Now it is not difficult to verify that the function (9) satisfy the differential equation (5).

Indeed by using the formulas (11) and (12) and also the fact that the function $\sum_{k=0}^{n-1} b_k g^{(k)}(t-t_0)$ a solution for the equation Ly=0 on obtain

$$a_n\left[\int_{t_0}^{t} u(\tau)g^{(n)}(t-\tau)d\tau + \frac{1}{a_n}u(t) + \sum_{k=0}^{n-1} b_k g^{(k+n)}(t-t_0)\right] +$$

$$a_{n-1}\left[\int_{t_0}^{t} u(\tau)g^{(n-1)}(t-\tau)d\tau + \sum_{k=0}^{n-1} b_k g^{(k+n-1)}(t-t_0)\right] + \ldots +$$

$$+ a_0\left[\int_{t_0}^{t} u(\tau)g(t-\tau)d\tau + \sum_{k=0}^{n-1} b_k g^{(k)}(t-t_0)\right] = u(t) +$$

$$+\left[a_n\int_{t_0}^{t} u(\tau)g^{(n)}(t-\tau)d\tau + \ldots + a_0\int_{t_0}^{t} u(\tau)g(t-\tau)\right] +$$

$$+ a_n\left[\sum_{k=0}^{n-1} b_k g^{(k)}(t-t_0)\right]^{(n)} + \ldots + a_0\left[\sum_{k=0}^{n-1} b_k g^{(k)}(t-t_0)\right] =$$

$$= u(t) + \int_{t_0}^{t} u(\tau)\left[a_n g^{(n)}(t-\tau) + \ldots + a_0 g(t-\tau)\right]d\tau + 0$$

$$= u(t) + \int_{t_0}^{t} u(\tau)\cdot 0 d\tau = u(t)$$

for $t > t_0$ .It is also easy to see that the function (9) satisfy the boundary value conditions

$$\lim_{t\to t_0} y^{(j)}(t) = y_j,\ j = 0,\ldots,n-1$$

For example

$$\lim_{t\to t_0}\int_{t_0}^{t} u(\tau)g(t-\tau)d\tau + \lim_{t\to t_0}\sum_{k=0}^{n-1} b_k g^{(k)}(t-t_0) = 0 + b_{n-1}g^{(n-1)}(0) =$$

$$= \frac{1}{a_n}\sum_{j=0}^{n-1-(n-1)} a_{n-1+1+j}y_j = \frac{1}{a_n}a_n y_0 = y_0$$

From theorem of Picard it result that the function (9) is the unique solution of the differential equation (5) for $t > t_0$

In the same way we can consider the equation

$$L\delta * Y = U, U \in D_{+}^{'} .$$

Because $Y_i = U + gH$ is a solution for (11), then every solution is given by $Y = y_h + U * gH$, where y_h is the general solution of the equation

$$L\delta * Y = 0$$

Since the distribution $U * gH \in D_{+}^{'}$ therefore supp $U * gH \subset [t_1, \infty)$. Conse quently on the interval $(-\infty, t_1)$, U*gH=0, that's on this interval $Y = y_h$ and we can in the point $t_0 = t_1 - \varepsilon$ to sets $y^{(k)}(t_0) = y_k$.
Because the solution is unique it follows that

$$Y = \sum_{k=0}^{n-1} b_k \left[g^{(k)}(t - t_1 + \varepsilon)\right] + U * [gH]$$

is the solution that we seek.

ЕДЕН ДОКАЗ НА ТЕОРЕМА ОД ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ

Никола Речкоски, с. Велгошти Охрид

Во оваа работа е даден доказ на проблемот на Пикар-Линделеф за обични диференцијални равенки со константни коефициенти.

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