On a Numerical Solution of a class of Sturm-Liouville Problems

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Abstract

A class of Sturm-Liouville problems containing spectral parameter in the boundary or interface conditions and coefficients which are piecewise functions are considered. Approximation of spectral problems using finite difference method and the estimates for the eigenvalues and eigenfunctions is given. Numerical solutions for these problems are obtained.

1 Introduction

Let us consider the initial boundary value problem for the heat equation with concentrated capacity at the interior point $x = \xi$ [4, 5, 8]

$$[c(x) + K\,\delta(x-\xi)]\,\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) + f(x,t), \quad (x,t) \in (0,\,1)\times(0,\,T), \tag{1.1}$$

$$u(0,\,t) = 0, \quad u(1,\,t) = 0, \quad 0 < t < T, \tag{1.2}$$

$$u(x,\,0) = u_0(x), \quad x \in (0,\,1), \tag{1.3}$$

where $K > 0$, $0 < c_1 \le a(x) \le c_2$, $0 < c_3 \le c(x) \le c_4$ and $\delta(x)$ is the Dirac distribution. It follows from (1.1), that the solution of this problem satisfies at $(x,t) \in Q_1 = (0,\,\xi)\times(0,\,T)$ and $(x,t) \in Q_2 = (\xi,\,1)\times(0,\,T)$ the equation

$$c(x)\,\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) + f(x,t)\,,$$

and at $x = \xi$ the conditions of conjugation

$$[u]_{x=\xi} \equiv u(\xi+0,t) - u(\xi-0,t) = 0\,, \qquad \left[a\,\frac{\partial u}{\partial x}\right]_{x=\xi} = K\,\frac{\partial u(\xi,t)}{\partial t}\,.$$

It is easy to see that the initial boundary value problem (1.1)–(1.3) can be written as an abstract Cauchy problem

$$B\frac{du}{dt} + Au = f(t), \quad t \in (0,T); \quad u(0) = u_0 \tag{1.4}$$

letting $H = L_2(0,1)$, $Au = -\frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big)$ and $Bu = [c(x) + K\,\delta(x - \xi)]\,u(x,\,t)$. Then $H_A = \overset{\circ}{W}{}_2^1(0,1)$,

$$\begin{aligned} \|w\|_A^2 &= \int_0^1 a(x)\,[w'(x)]^2\,dx, \\ \|w\|_B^2 &= \int_0^1 c(x)\,w^2(x)\,dx + K\,w^2(\xi)\,. \end{aligned}$$

Thus, the following spectral problem can be obtained:

$$\begin{gathered} (a(x)w')' + \lambda c(x)w = 0, \quad x \in (0,\xi) \cup (\xi,1), \\ [w]_{x=\xi} = w(\xi+0) - w(\xi-0) = 0, \quad [aw']_{x=\xi} + \lambda K w(\xi) = 0, \\ w(0) = w(1) = 0. \end{gathered} \tag{1.5}$$

Further we assume that the function $c(x)$ is continuous on $[0,1]$ and $a(x)$ has finite jump in the point $x = \xi$.

This spectral problem can be written in the form

$$Aw = \tilde{\lambda} Bw \tag{1.6}$$

with A, B as defined above. In such a way, the spectrum of (1.6) is discrete, all eigenvalues are positive, $\tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots$, $\tilde{\lambda}_n \to \infty$, while the eigenfunctions $w = w_n$, $n = 1, 2, \ldots$ satisfy the condition of orthogonality

$$(w_j, w_k)_B = (Bw_j, w_k) = (\tilde{w}_j, \tilde{w}_k) = \delta_{jk}$$

and represent the basis of the space H_B.

The solution of problem (1.4) can be written in the form

$$u(t) = \sum_{n=1}^{\infty} e^{-\tilde{\lambda}_n t}\left[c_n + \int_0^t e^{\tilde{\lambda}_n \tau} f_n(\tau)d\tau\right] w_n,$$

where

$$c_n = (u_0, w_n)_B, \quad f_n(t) = (f(t), w_n).$$

2 Sturm–Liouville Problem

We consider a Sturm-Liouville problem:

$$(p(x)v')' - q(x)v + \lambda r(x)v = 0\,, \tag{2.1}$$

where λ is the eigenvalue, $v(x)$ is the eigenfunction, $p(x)$, $q(x)$, $r(x)$ are piecewise continuous functions on $[0,1]$ such that

$$0 < c_1 \le p(x) \le c_2\,, \quad 0 < c_1 \le r(x) \le c_3\,, \quad 0 \le q(x) \le c_4\,, \tag{2.2}$$

$c_1, c_2, c_3, c_4 = \text{const.}$

Let ξ be an interior point in $(0\,,1)$ where $v(x)$ has to satisfy the condition of conjugation

$$[v]_{x=\xi} = v(\xi+0) - v(\xi-0) = 0\,, \quad [pv']_{x=\xi} = -\lambda K v(\xi),\ K = \text{const.} > 0\,. \tag{2.3}$$

Then $p(x)$ could have a discontinuity of first order at the point $x = \xi$, $(0 < \xi < 1)$. The boundary conditions could also consist spectral parameter

$$\begin{aligned} -\quad & \alpha_0 v'(0) + \beta_0 v(0) = \lambda\gamma_0 v(0)\,, \quad \alpha_0 + \beta_0 > 0,\ \alpha_0\,, \beta_0 \ge 0\,, \\ & \alpha_1 v'(1) + \beta_1 v(1) = \lambda\gamma_1 v(1)\,, \quad \alpha_1 + \beta_1 > 0,\ \alpha_1\,, \beta_1 \ge 0\,, \\ & \alpha_i,\ \beta_i,\ \gamma_i = \text{const.}\,,\ i = 1, 2\,. \end{aligned} \tag{2.4}$$

Using Dirac distribution, the problem (2.1), along with the conditions (2.3), could be written in the following form :

$$(p(x)v')' - q(x)v + \lambda[r(x) + K\delta(x-\xi)]v = 0\,, \tag{2.5}$$

or in the operator form $Av = \lambda Bv$, letting $H = L_2(0,1)$ and

$$Av = -(p(x)v')' + q(x)v\,, \quad Bv = [r(x) + K\delta(x-\xi)]v\,. \tag{2.6}$$

This kind of spectral problems appears, as it was shown previously , while solving the heat equation with concentrated capacity and combinations of various boundary conditions as a result of using the method of separation of variables [4, 1, 9].

As a model we shall consider the equation

$$[1 + K\delta(x-\xi)]\frac{\partial u}{\partial t} = p\frac{\partial^2 u}{\partial x^2} - qu, \quad p,\ q = \text{const.} > 0 \tag{2.7}$$

with boundary conditions: $u(0,t)=0,\quad \frac{\partial u}{\partial x}(1,t)=0,$ and initial value $u(x,0) = u_0(x)$. As it was previously shown, following spectral problem can be obtained

$$-\frac{d^2w}{dx^2} = \lambda w(x),\; x\in(0,\xi)\cup(\xi,1)$$

$$w(0)=0,\; w'(1)=0,$$

$$[w]_\xi = 0,\quad -\left[\frac{dw}{dx}\right]_\xi = \lambda K w(\xi).$$

This problem is a special case of the problem (2.1)–(2.4) setting in the boundary conditions (2.4)

$$\alpha_0 = \beta_1 = \gamma_0 = \gamma_1 = 0\,,\quad \alpha_1 = p(1)\,,\quad \beta_0 = 1\,. \tag{2.8}$$

It can be proved that for such problem the following assertions hold [2, 3].

Theorem 1 *The Sturm-Liouville problem (2.1)–(2.4) with coefficients given in (2.8) has a countable set of eigenvalues* $0<\lambda_1<\lambda_2<\cdots$ *to which correspond the eigenfunctions* $v_1(x)$, $v_2(x),\ldots.$ *The eigenfunctions* $\{v_n(x)\}$ *form a complete orthogonal system with respect to a norm* $\|\cdot\|_B$ *which arises from the inner product* $(\cdot,\cdot)_B$*, where*

$$(u,v)_B = \int_0^1 r(x)u\,v\,dx + Ku(\xi)v(\xi)\,. \tag{2.9}$$

Theorem 2 *The eigenvalues of the problem (2.1)–(2.4) with coefficients (2.8),* $\lambda_n\to\infty$, $n\to\infty$, *satisfy the inequalities:*

$$c_5 n^2 \le \lambda_n \le c_6 n^2\,,\quad c_5>0\,, \tag{2.10}$$

c_5 *and* c_6 *are independent of* c_i, $i=1,2,3,4$ *and* n.

Theorem 3 *The eigenfunctions of the problem (2.1)–(2.4) and their derivatives satisfy the inequalities*

$$|v_n(x)|\le c_7\,,\quad |v_n'(x)|\le c_8'\sqrt{\lambda_n}\le c_8 n\,, \tag{2.11}$$

where c_7, c_8 *are positive constants independent of* n.

The solution of spectral problem given above can be written in explicit form

$$w(x) = \begin{cases} A \sin\, \alpha x, & x \in (0, \xi) \\ B \cos\, \alpha(1-x), & x \in (\xi, 1) \end{cases}$$

We obtain the values of the constants A and B using the first condition of conjugation: $A = C cos\, \alpha(1-\xi)$, $B = C sin\, \alpha\xi$, where C is a multiplicative constant, and we can set $C = 1$. The values of $\alpha = \alpha_n$, $n = 1, 2, \ldots$ are the roots of the transcendental equation

$$\alpha = \frac{1}{K}[ctg\, \alpha\xi - tg\, \alpha(1-\xi)].$$

There exists a countable set of solutions $\alpha = \alpha_n, n = 1, 2, \ldots$ of this equation. Thus we can obtain the eigenvalues $\lambda = \lambda_n = \alpha_n^2, \quad 0 < \lambda_1 < \lambda_2 \cdots, \ \lambda_n \to \infty$, and respective eigenfunctions $w = w_n(x)$, $n = 1, 2, \ldots$.

There is another class of solution $w(x) = \sin \alpha x$ which exists if $\xi = \dfrac{2m}{2k+1}$, k, $m \in N$. Then for $n = 0, 1, 2, \ldots$

$$\lambda_n = p\left[(2k+1)(2n+1)\frac{\pi}{2}\right]^2 + q, \quad w_n = \sin(2k+1)(2n+1)\frac{\pi}{2}x\,.$$

3 Difference Sturm-Liouville Problems

Let $\overline{\omega}_h = \{x_i = ih,\ i = 0, 1, \ldots, N,\ hN = 1\}$ is a uniform mesh in $[0, 1]$ chosen so that $\xi = x_m$ is a node. We approximate the problem (2.1)–(2.4) with coefficients (2.8) on the mesh $\overline{\omega}_h$ by the difference scheme

$$\begin{gathered} -(ay_{\overline{x}})_x + dy = \lambda^h\left(\rho(x) + K\delta_h(x-\xi)\right)y\,, \quad x \in \omega_h\,, \\ y_0 = 0\,, \qquad ay_{\overline{x}} + \frac{h}{2}dy = \frac{h}{2}\lambda^h\rho y\,, \quad x = x_N\,, \end{gathered} \tag{3.1}$$

where

$$\delta_h(x-\xi) = \begin{cases} 0\,, & x \in \omega_h \backslash \{\xi\} \\ \dfrac{1}{h}, & x = \xi \end{cases}\,,$$

or

$$-(ay_{\overline{x}})_x + dy = \lambda^h \rho\, y\,, \qquad 0 < x = x_i = ih < 1\,,\ i \neq m\,,$$

$$-\frac{h}{\overline{K}}(ay_{\overline{x}})_x + \frac{h}{\overline{K}}\overline{d}\, y = \lambda^h y\,, \qquad x = x_m\,,\ \overline{K} = K + h\rho(\xi)\,,$$

$$\overline{d} = \begin{cases} \dfrac{d(x_m - 0) + d(x_m + 0)}{2}\,, & x = x_m = \xi \\ d\,, & x \in \omega_h \setminus \{\xi\} \end{cases}\,,$$

$$y_0 = 0\,, \qquad ay_{\overline{x}} + \frac{h}{2}dy = \frac{h}{2}\lambda^h \rho y\,, \qquad x = x_N\,.$$

We shall denote by $\Lambda\, y$ difference operator

$$\Lambda\, y = \begin{cases} - & (ay_{\overline{x}})_x + dy\,, & x = x_i,\ i = 1,\, 2,\, \dots\,, N-1 \\ & \dfrac{2}{h} ay_{\overline{x}} + dy\,, & i = N \\ & 0\,, & i = 0 \end{cases}$$

while $d(x_i) = q(x_i)\,, \quad \rho(x_i) = r(x_i)\,,$

$$a(x) = \frac{p(x) + p(x-h)}{2}\,, \qquad \text{for } x \neq \xi, \xi + h\,,$$

$$a(\xi) = \frac{p(\xi - 0) + p(\xi - h)}{2}\,, \quad a(\xi + h) = \frac{p(\xi + h) + p(\xi + 0)}{2}\,.$$

Thus the following inequalities hold:

$$0 < c_1 \leq a \leq c_2\,, \quad 0 < c_1 \leq \rho(x) \leq c_3\,, \quad 0 \leq d(x) \leq c_4\,. \qquad (3.2)$$

Now our goal is to find nontrivial solutions of the problem (3.1) (the eigenfunctions) which correspond to the values of the parameter λ^h (the eigenvalues). It is already prove that [3]:

Theorem 4 *There exists $N-1$ eigenvalues of the problem (3.1), $0 < \lambda_1^h < \dots < \lambda_{N-1}^h$ to which correspond the eigenfunctions $y_1(x), \dots, y_{N-1}(x)$. The eigenfunctions $\{y_n(x)\}$ form an orthogonal system in the l_{N-1}^2 space with scalar product*

$$(y, v]_{B_h} = \sum_{\substack{i=1 \\ i \neq m}}^{N-1} \rho_i\, y_i\, v_i\, h + \overline{K} y_m\, v_m + \frac{h}{2}\rho_N\, y_N\, v_N\,, \quad \overline{K} = K + h\rho(\xi)\,. \quad (3.3)$$

Theorem 5 *The eigenvalues of the problem (3.1) satisfy inequalities*

$$M_1' n^2 \le \lambda_n^h \le M_2' n^2 \,, \quad n = 1, 2, \ldots, N-1 \,, \tag{3.4}$$

where M_1' and M_2' are positive constants independent of h and n.

Theorem 6 *For the eigenfunctions of the problem (3.1) the following estimates hold*

$$\|y_n\|_C \le M_1 \sqrt{n} \,, \quad \|(y_n)_{\overline{x}}\|_C \le M_2 n^{3/2} \,, \tag{3.5}$$

where $\|y\|_C = \max_{x \in \omega_h} |y(x)| \,, \; \|y_{\overline{x}}\|_C = \max_{1 \le i \le N} |y_{\overline{x},i}|$, M_1, M_2 are constants independent of h and n.

It can be proved that the scheme (3.1) has a second order of accuracy ($\|y_n - u_n\|_C = O(h^2)$, $|\lambda_n^h - \lambda_n| = O(h^2)$) except at the point $x = \xi$ where $H_N[u] - H[u] = O(h)$, i.e. $\|y_n - u_n\|_C = O(h)$ [2].

4 Numerical Experiments

In order to approve theoretical results given above, we made some numerical experiments using program package MatLab.We set $p = 1$, $q = 0$ in (2.7) and use difference scheme (3.1), uniform mesh where the number of nodes is $2^k \cdot 10$, $k = 0, 1, \ldots, 5$. In the table 1 the approximation error of the first four eigenvalues is presented. It is obvious that approximation error has the lowest value for the first eigenvalue and it increases for every next eigenvalue. On the other hand, the approximation error depends on the number of nodes N, so we can see that the increasing of the value of N implies decreasing of the error.

$N \setminus$ Err	$\|\lambda_1 - \lambda_1^h\|$	$\|\lambda_2 - \lambda_2^h\|$	$\|\lambda_3 - \lambda_3^h\|$	$\|\lambda_4 - \lambda_4^h\|$
10	1.74 E-03	1.76 E-01	1.65 E-00	7.19 E-00
20	4.36 E-04	4.42 E-02	4.20 E-01	1.85 E-00
40	1.09 E-04	1.11 E-02	1.05 E-01	4.65 E-01
80	2.73 E-05	2.77 E-03	2.64 E-02	1.16 E-01
160	6.82 E-06	6.93 E-04	6.59 E-03	2.91 E-02
320	1.70 E-06	1.73 E-04	1.65 E-03	7.28 E-03

Table 1

In a table 2 the convergence rate calculated by the formula

$$\rho_N = \log_2 \frac{\|u - v\|_{\infty,N}}{\|u - v\|_{\infty,2N}},$$

is given. Here, we denote by u an exact and by v approximate values. We can see from the table that convergence rate of the eigenvalues is near to 2, like it was proved in previous paragraph.

$N \setminus \rho_N$	$\rho_N(\lambda_1)$	$\rho_N(\lambda_2)$	$\rho_N(\lambda_3)$	$\rho_N(\lambda_4)$
ρ_{10}	1.99902	1.99238	1.98037	1.96213
ρ_{20}	1.99976	1.99809	1.99509	1.99051
ρ_{40}	1.99994	1.99952	1.99877	1.99763
ρ_{80}	1.99998	1.99988	1.99969	1.99941
ρ_{160}	2.00001	1.99997	1.99992	1.99985

Table 2

For the eigenfunctions similar discussion can be done. Approximation error

$$\mathrm{Err}_i = \max_{x\in\bar{\omega}} |y_i^h(x) - u_i(x)|,$$

for $i = 1, 2, 3, 4$ is given in the table 3, while the values of convergence rate are presented in table 4.

$N \setminus \mathrm{Err}_i$	Err_1	Err_2	Err_3	Err_4
10	1.79 E-05	1.30 E-03	3.12 E-03	4.40 E-03
20	3.18 E-06	2.41 E-04	5.57 E-04	8.14 E-04
40	5.61 E-07	4.37 E-05	9.85 E-05	1.48 E-04
80	9.90 E-08	7.82 E-06	1.74 E-05	2.65 E-05
160	1.75 E-08	1.39 E-06	3.09 E-06	4.72 E-06
320	3.09 E-09	2.47 E-07	5.46 E-07	8.36 E-07

Table 3

$N \setminus \rho_N$	$\rho_N(y_1)$	$\rho_N(y_2)$	$\rho_N(y_3)$	$\rho_N(y_4)$
ρ_{10}	2.49591	2.43613	2.48729	2.43393
ρ_{20}	2.50218	2.46487	2.49811	2.45766
ρ_{40}	2.50226	2.48128	2.49726	2.48314
ρ_{80}	2.50144	2.49114	2.49893	2.49006
ρ_{160}	2.50079	2.49549	2.49962	2.49544

Table 4

References

[1] Belinskiy, B. P., Dauer, J. F.: *Eigenoscillations of mechanical systems with boundary conditions containing the frequency*, Q.uarterly of Appl.Math. 16 (1999), 521–541.

[2] Gegovska-Zajkova S. : *Numerical solution of Sturm-Liouville problems containing spectral parameter in boundary or interface conditions*, Matematichki bilten, Skopje (to appear).

[3] Геговска-Зајкова С. : *Спектрални проблеми кои содржат сингуларни дистрибуции и нивна примена*, Докторска дисертација, Скопје 2004.

[4] Jovanovic B. S., Vulkov L. G.: *On the convergence of finite difference schemes for the heat equation with concentrated capacity*, Numer. Math. 89, No 4 (2001), 715–734.

[5] Jovanovic B. S., Vulkov L. G., Gegovska-Zajkova S.: *On the convergence of finite difference schemes for the heat equation with concentrated capacity*, Numer. Math. 89, No 4 (2001), 715–734.

[6] Марчук, Г. И., Агошков, В. И.: *Введение в проекционно сеточные методы* , Наука, Москва (1981).

[7] Михайлов Б. П.: *Дифференциальные уравнения в частных производных*, Наука, Москва (1976).

[8] Samarskii A. A. : *Theory of difference schemes*. Nauka, Moscow 1989 (in Russian).

[9] Tihonov A. N., Samarskii A. A.: *Equations of Mathematical Physics*, GITTL, Moscow 1953 (in Russian).